On the geometry of curves and conformal geodesics in the Möbius space

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Abstract

This paper deals with the study of some properties of immersed curves in the conformal sphere $\mathbb{Q}_n$, viewed as a homogeneous space under the action of the Möbius group. After an overview on general well-known facts, we briefly focus on the links between Euclidean and conformal curvatures, in the spirit of F. Klein's Erlangen program. The core of the paper is the study of conformal geodesics, defined as the critical points of the conformal arclength functional. After writing down their Euler-Lagrange equations for any n, we prove an interesting codimension reduction, namely that every conformal geodesic in $\mathbb{Q}_n$ lies, in fact, in a totally umbilical 4-sphere $\mathbb{Q}_4$. We then extend and complete the work in [20] by solving the Euler-Lagrange equations for the curvatures and by providing an explicit expression even for those conformal geodesics not included in any conformal 3-sphere.

1 Introduction

The investigation of the conformal properties of submanifolds of the unit n-dimensional sphere is a well-developed field in differential geometry. Its deep links range, for example, from the classical theory of curves and surfaces in $\mathbb{R}^3$ to the theory of integrable systems, general relativity and so on. Among the various approaches that have been used in the study of the subject, Cartan's method of the moving frame (see [17], [28] and the original books of E. Cartan, [11], [10]) stands out for its usefulness, depth and generality. In particular, the theory of homogeneous spaces, which encompasses the conformal geometry of the sphere, gives a vast field where this technique applies at best. In this paper, we use Cartan's method to deal with the geometry of immersed curves. In particular, our main concern is to complete the characterization of conformal geodesics $f : I \subset \mathbb{R} \to \mathbb{Q}_n$, begun with the work of E. Musso in [20] in the case of $\mathbb{Q}_3$. Such curves arise as the stationary points of the integral of a conformally invariant 1-form, $\mathrm{d}s$, called

the conformal arclength. Roughly speaking, ds is the conformal analogue of the differential of the Euclidean arclength, and is linked to it through some classical formulas in [19], [29], [33], [35].
Using Griffiths' formalism (see [15]), in [20] the author wrote the Euler-Lagrange equations for $n = 3$, obtaining the following system of ODEs for the conformal curvatures $\mu_1, \mu_2 \in C^\infty(I)$:

$$\begin{cases} \dot{\mu}_1 + 3\mu_2\dot{\mu}_2 = 0 \\ \ddot{\mu}_2 = \mu_2^3 + 2\mu_1\mu_2, \end{cases} \tag{1}$$

where the dot denotes the derivative with respect to the arclength parameter s; $\{\mu_1, \mu_2\}$ constitutes a complete set of conformal invariants which characterize f up to a conformal motion of $\mathbb{Q}_3$. He then solved (1) by means of elliptic functions, and found the explicit form of every conformal geodesic. His method leads to a lengthy computation as the dimension n grows, thus the need of a different approach for a generic n. In this paper (Section 5), we obtain a simple form for the Euler-Lagrange equations in any dimension without the aid of Griffiths' formalism. This expression leads to the following, unexpected, degeneracy result (Theorem 5.5):

Theorem. *Every conformal geodesic $f : I \subset \mathbb{R} \to \mathbb{Q}_n$ lies in some totally umbilical 4-sphere $\mathbb{Q}_4 \subset \mathbb{Q}_n$.*

This allows us to limit ourselves to the study of conformal geodesics in $\mathbb{Q}_4$. After characterizing the degenerate cases, we concentrate on generic conformal geodesics in the 4-sphere, obtaining the Euler-Lagrange equations

$$\begin{cases} \dot{\mu}_1 + 3\mu_2\dot{\mu}_2 = 0, \qquad \mu_2 > 0, \\ \ddot{\mu}_2 = \mu_2^3 + 2\mu_1\mu_2 + \mu_2\mu_3^2, \qquad \mu_3 > 0, \\ 2\dot{\mu}_2\mu_3 + \mu_2\dot{\mu}_3 = 0. \end{cases} \tag{2}$$

relating the conformal curvatures $\{\mu_1, \mu_2, \mu_3\}$ of a conformal geodesic. As it is apparent, system (2) exhibits a clear similarity with (1) and can be integrated by using elliptic functions (Section 6). This suggests that an explicit integration of the equations of motion can be achieved even for $n = 4$. As a matter of fact, we obtain an explicit expression for f up to a conformal motion, see the discussion after Proposition 6.1. Surprisingly, this process reveals even easier than the corresponding one for $n = 3$, since a unique case has to be analyzed, differently from [20] where the author has to deal with three distinct cases.

We need to introduce some background material for our purposes. Although part of it is quite standard, for the convenience of the reader and to keep the paper basically self-contained and the notation coherent, we have decided to begin with a brief account on the basic results in the conformal geometry of curves. The first two sections therefore deal with the homogeneous space structure of the n-sphere (Section 2) and the geometry of immersed curves (Section 3). Our approach follows closely that of [27] and [29] except for a slightly different notation. Among classical and modern references, we mainly recommend [16] and [28]: they both present a complete, self-contained account on the conformal group and two different, very nice geometrical proofs of Liouville's theorem. An elementary introduction to the subject is also given in [4], and standard further references are [36], [13], [5]. For subtle topological details, we suggest to the reader [22]. In Section 4 we discuss the links between Euclidean and conformal invariants. Our approach leads to a (slight) improvement of a result in [1] by showing that their set of conformal invariants indeed coincides with the one presented here, see Proposition 4.1. Another application of the circle of ideas of Section 4 is a natural proof of the conformal invariance of the so-called total twist of a closed curve in $\mathbb{Q}_3$, that is, the quantity

$$\mathrm{Tw}(f) = \frac{1}{2\pi}\int_I \tau \mathrm{d}s_e \qquad (\mathrm{mod}\ \mathbb{Z}),$$

where τ is the Euclidean torsion of f and s_e is the Euclidean arclength. This result has been proved in [8], but it seems that their elegant argument, although not far from our approach, is based on some sort of "magical" identity involving a globally defined angle, whose appearing seems to us not completely justified.

2 Construction of the conformal sphere $\mathbb{Q}_n$: a short review

Consider $\mathbb{S}^n$ and $\mathbb{R}^n$ with their standard metrics of constant curvatures, and let $\sigma : \mathbb{S}^n \backslash \{N\} \to \mathbb{R}^n$ be the stereographic projection, where $N = (1, 0, \dots, 0) \in \mathbb{R}^{n+1}$ is the north pole. It is well known that σ is a conformal diffeomorphism. If $n \geq 3$, by Liouville's theorem ([12], pp.138-141; [16], pp.52-53, [28], pp. 289-290), every conformal diffeomorphism of $\mathbb{S}^n$ is of the form $\sigma^{-1} \circ g \circ \sigma$, where g is a composition of Euclidean similarities of $\mathbb{R}^n$ with possibly the inversion $\mathbb{R}^n \backslash \{0\} \ni x \mapsto x/|x|^2$. The assertion holds even for $n = 2$, although a proof of this fact relies, for instance, on standard compact Riemann surfaces theory since Liouville's theorem is false for $\mathbb{C}$. We observe that the conformal group $\mathrm{Conf}(\mathbb{S}^2)$ can be also identified with the fractional linear transformations of $\mathbb{C}$, either holomorphic or anti-holomorphic. From now on, we let $n \geq 2$ and we fix the index convention $1 \leq A, B, C \leq n$. We denote with $\mathbb{Q}_n$ the Darboux hyperquadric

$$\mathbb{Q}_n = \left\{ (x^0 : x^A : x^{n+1}) \mid \sum_A (x^A)^2 - 2x^0 x^{n+1} = 0 \right\} \subset \mathbb{P}^{n+1}(\mathbb{R}).$$

The Dirac-Weyl embedding $\chi : \mathbb{R}^n \to \mathbb{Q}_n$ is defined by

$$\chi : x \longmapsto \left(1 : x : \frac{1}{2}|x|^2\right)$$

and it extends to a diffeomorphism $\chi \circ \sigma : \mathbb{S}^n \to \mathbb{Q}_n$ by setting $\chi \circ \sigma(N) = (0 : 0 : 1)$. The advantage of such a representation for the sphere is that every conformal diffeomorphism of $\mathbb{S}^n$ acts as a linear transformation on the homogeneous coordinates of $\mathbb{Q}_n$, so that $\mathrm{Conf}(\mathbb{S}^n)$ can be viewed as the projectivized of the linear subgroup of $GL(n+2)$ preserving the quadratic form which defines the Darboux hyperquadric.
Endow $\mathbb{R}^{n+2}$ with the Lorentzian metric $\langle\, ,\rangle$ represented, with respect to the standard basis $\{\eta_0, \eta_A, \eta_{n+1}\}$, by the matrix

$$S = \begin{pmatrix} 0 & 0 & -1 \\ 0 & I_n & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

and let L^+ be the positive light cone, that is, $L^+ = \{v = {}^t(v^0, v^A, v^{n+1}) \in \mathbb{R}^{n+2} : {}^tvSv = 0,\ v^0 + v^{n+1} > 0\}$. Note that L^+ projectivizes to $\mathbb{Q}_n$ and that $\eta_0, \eta_{n+1} \in L^+$. Moreover, there is a bijection between $\mathrm{Conf}(\mathbb{S}^n)$ and the Lorentz group of $\langle\, ,\rangle$ preserving the positive light cone (usually called the orthochronous Lorentz group). This gives a Lie group structure to the conformal group $\mathrm{Conf}(\mathbb{S}^n)$, which can be proved to be unique when the action of $\mathrm{Conf}(\mathbb{S}^n)$ on $\mathbb{S}^n$ is required to be smooth (see [22], pp. 95-98). In particular, the identity component of the Lorentz group is called the **Möbius group**, $\mathrm{Möb}(n)$, and coincides with the subgroup of the orientation preserving elements of $\mathrm{Conf}(\mathbb{S}^n)$. The transitivity of the action of $\mathrm{Möb}(n)$ on the n-sphere gives $\mathbb{Q}_n$ a homogeneous space structure, allowing us to identify it with the space of left cosets $\mathrm{Möb}(n)/\mathrm{Möb}(n)_0$, where $\mathrm{Möb}(n)_0$ is the isotropy subgroup of $[\eta_0] \in \mathbb{Q}_n$:

$$\mathrm{Möb}(n)_0 = \left\{ \begin{pmatrix} r^{-1} & {}^txA & \frac{1}{2}r|x|^2 \\ 0 & A & rx \\ 0 & 0 & r \end{pmatrix} \,\middle|\, \begin{array}{l} r > 0,\ x \in \mathbb{R}^n, \\ A \in SO(n) \end{array} \right\}. \tag{3}$$

It follows that the principal bundle projection $\pi : \mathrm{Möb}(n) \to \mathbb{Q}_n$ associates to a matrix $G = (g_0|g_A|g_{n+1})$ the point $[G\eta_0] = [g_0] \in \mathbb{Q}_n$. From now on, we shall use the Einstein summation convention. Let $\mathfrak{möb}(n)$ denote the Lie algebra of $\mathrm{Möb}(n)$; the Maurer-Cartan form Φ of $\mathrm{Möb}(n)$ is the $\mathfrak{möb}(n)$-valued 1-form

$$\Phi = \begin{pmatrix} \Phi^0_0 & \Phi^0_B & 0 \\ \Phi^A_0 & \Phi^A_B & \Phi^A_{n+1} \\ 0 & \Phi^{n+1}_B & \Phi^{n+1}_{n+1} \end{pmatrix},$$

with the symmetry relations

$$\Phi^{n+1}_{n+1} = -\Phi^0_0, \quad \Phi^A_B = -\Phi^B_A, \quad \Phi^A_{n+1} = \Phi^0_A, \quad \Phi^{n+1}_B = \Phi^B_0$$

and satisfying the structure equation $\mathrm{d}\Phi + \Phi \wedge \Phi = 0$, which component-wise reads

$$\begin{cases} \mathrm{d}\Phi^0_0 &= -\Phi^0_A \wedge \Phi^A_0; \\ \mathrm{d}\Phi^A_0 &= -\Phi^A_0 \wedge \Phi^0_0 - \Phi^A_B \wedge \Phi^B_0; \\ \mathrm{d}\Phi^0_A &= -\Phi^0_0 \wedge \Phi^0_A - \Phi^0_B \wedge \Phi^B_A; \\ \mathrm{d}\Phi^A_B &= -\Phi^A_0 \wedge \Phi^0_B - \Phi^A_C \wedge \Phi^C_B - \Phi^0_A \wedge \Phi^B_0. \end{cases} \tag{4}$$

Through a local section $s : U \subset \mathbb{Q}_n \to \mathrm{M\ddot{o}b}(n)$, Φ pulls back to a flat Cartan connection $\psi = s^*\Phi = s^{-1}\mathrm{d}s$. In particular, the set $\{\psi^A_0\}$ gives a local basis for the cotangent bundle of $\mathbb{Q}_n$. Under a change of section $\widetilde{s} = sK$, where $K : U \subset \mathbb{Q}_n \to \mathrm{M\ddot{o}b}(n)_0$, the change of gauge becomes

$$\widetilde{\phi} = \widetilde{s}^{-1}\mathrm{d}\widetilde{s} = K^{-1}\psi K + K^{-1}\mathrm{d}K.$$

By the expression of $\mathrm{M\ddot{o}b}(n)_0$ in (3), we have in particular

$$(\widetilde{\psi}^A_0) = r^{-1}\, {}^t\!A(\psi^A_0), \tag{5}$$

where (ψ^A_0) stands for the column vector whose A-th component is ψ^A_0. It follows that

$$\widetilde{\psi}^A_0 \otimes \widetilde{\psi}^A_0 = r^{-2}\psi^A_0 \otimes \psi^A_0, \qquad \widetilde{\psi}^1_0 \wedge \ldots \wedge \widetilde{\psi}^n_0 = r^{-n}\psi^1_0 \wedge \ldots \wedge \psi^n_0,$$

which implies that

$$\Big\{ \big(U, \psi^A_0 \otimes \psi^A_0\big) \;:\; U \subset \mathbb{Q}_n \text{ domain of a local section } s : U \to \mathrm{M\ddot{o}b}(n) \Big\}$$

defines a conformal structure on $\mathbb{Q}_n$, that is, a collection of locally defined metrics varying conformally on the intersection of their domains of definition, together with an orientation (locally defined by $\psi^1_0 \wedge \ldots \wedge \psi^n_0$), both preserved by $\mathrm{M\ddot{o}b}(n)$. It is easy to prove that, with this conformal structure, $\chi \circ \sigma : \mathbb{S}^n \to \mathbb{Q}_n$ is a conformal diffeomorphism. This gives sense to the whole construction.

3 The Frenet-Serret equations for curves in $\mathbb{Q}_n$

Let $I \subset \mathbb{R}$ be an open interval and let $f : I \to \mathbb{Q}_n$ be an immersion. We give an outline of the frame reduction procedure and deduce the Frenet-Serret equation for the curve f. This is a standard procedure, see for example [29], [20]. Henceforth we adopt the following index conventions:

$$1 \le A, B, C, \ldots \le n, \qquad 2 \le \alpha, \beta, \ldots \le n.$$

Let $e : U \subset I \to \text{Möb}(n)$ be a zeroth order frame along f, namely a smooth map such that $\pi \circ e = f_{|U}$, and set $\phi = e^*\Phi$. If $\widetilde{e}$ is another zeroth order frame along f, then $\widetilde{e} = eK$, where K is a $\text{Möb}(n)_0$-valued smooth map. It follows that

$$\widetilde{\phi} = \widetilde{e}^*\Phi = \widetilde{e}^{-1}\mathrm{d}\widetilde{e} = K^{-1}\phi K + K^{-1}\mathrm{d}K. \tag{6}$$

From (5) and since f is an immersion, for a fixed point $p \in I$, it is always possible to consider a frame e such that

$$\phi_0^\alpha := e^*\Phi_0^\alpha = 0 \tag{7}$$

at p. The isotropy subgroup of such frames is

$$\text{Möb}(n)_1 = \left\{ \begin{pmatrix} r^{-1} & x & {}^tyB & \frac{1}{2}r(x^2+|y|^2) \\ 0 & 1 & 0 & rx \\ 0 & 0 & B & ry \\ 0 & 0 & 0 & r \end{pmatrix} \;\middle|\; \begin{array}{l} r > 0,\ x \in \mathbb{R} \\ y \in \mathbb{R}^{n-1}, \\ B \in SO(n-1) \end{array} \right\} \tag{8}$$

and since it is independent of p, by the standard theory of frame reduction (see [31], [32], [28]) smooth zeroth order frames can be chosen, which satisfy condition (7) in a suitable neighbourhood of p. Such frames will be called **first order frames**.

If e and $\widetilde{e}$ are first order frames, they are related by $\widetilde{e} = eK$, where now K takes values in $\text{Möb}(n)_1$. From (6) we get

$$\widetilde{\phi}_0^1 = r^{-1}\phi_0^1, \qquad \widetilde{\phi}_1^\alpha = B_\alpha^\beta\Big(\phi_1^\beta - y^\beta\phi_0^1\Big), \tag{9}$$

thus the form ϕ_0^1 determines a conformal structure on I. If we set $\phi_1^\alpha = h^\alpha\phi_0^1$, where h^α are smooth functions locally defined on I, then

$$\widetilde{h}^\alpha = rB_\alpha^\beta\Big(h^\beta - y^\beta\Big). \tag{10}$$

At any point $p \in I$, we can therefore consider a first order frame such that h^α, hence ϕ_1^α, vanishes. Since the isotropy subgroup preserving such frames is

$$\text{Möb}(n)_2 = \left\{ \begin{pmatrix} r^{-1} & x & 0 & \frac{1}{2}rx^2 \\ 0 & 1 & 0 & rx \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & r \end{pmatrix} \;\middle|\; \begin{array}{l} r > 0,\ x \in \mathbb{R} \\ B \in SO(n-1) \end{array} \right\}, \tag{11}$$

independent of p, we can locally choose a frame satisfying $h^\alpha = 0$. Such frames will be called **second order frames** or **Darboux frames** along f. Under a change of Darboux frames $\widetilde{e} = eK$, for a $\text{Möb}(n)_2$-valued K, denoting by e_a the a-th column of e, we have

$$\left\{ \begin{array}{rcl} \widetilde{\phi}_\beta^\alpha & = & B_\alpha^\gamma\phi_\delta^\gamma B_\beta^\delta + B_\alpha^\gamma\mathrm{d}B_\beta^\gamma \\ \widetilde{e}_\alpha & = & B_\alpha^\beta e_\beta. \end{array} \right. \tag{12}$$

Thus we can define a vector bundle, denoted by N and called the **normal bundle**, by declaring $\{e_\alpha\}$ a local basis. When $n = 2$, the normal bundle is simply the span of e_2. N is endowed with a global inner product, defined by requiring $\{e_\alpha\}$ to be orthonormal, and a compatible connection $\overline{\nabla}$ by setting

$$\overline{\nabla} e_\alpha = \phi_\alpha^\beta \otimes e_\beta.$$

We set

$$\phi_\alpha^0 = p^\alpha \phi_0^1, \tag{13}$$

for some smooth local functions p^α. Then, under a change of Darboux frames,

$$\widetilde{p}^\alpha = r^2 B_\alpha^\beta p^\beta, \tag{14}$$

which, together with (9) implies that the form

$$\sqrt[4]{\sum_\alpha (p^\alpha)^2 \phi_0^1} \tag{15}$$

is globally defined. Note that it may be only of class $C^{0,1/2}$ locally around the points where it vanishes. Since I is an interval, there exists a function $s : I \to \mathbb{R}$ of class $C^{1,1/2}(I)$ such that

$$\sqrt[4]{\sum_\alpha (p^\alpha)^2 \phi_0^1} = \mathrm{d}s. \tag{16}$$

Observe that s is defined up to a constant. Moreover, up to changing the sign, it is non-decreasing and smooth, strictly increasing on every connected subinterval of I where $\sum_\alpha (p^\alpha)^2 > 0$.

Definition 3.1. *Every function $s : I \to \mathbb{R}$ such that* (16) *holds is called a* **conformal arclength**.

Driven by geometrical considerations, we make the non-degeneracy assumption that ds never vanishes on I.

Definition 3.2. *Given an immersed curve $f : I \to \mathbb{Q}_n$, a point $q \in I$ is called* 1**-generic** *if*

$$\sum_\alpha (p^\alpha)^2 \neq 0 \tag{17}$$

at q. The immersion f is said to be 1**-generic** *if every point of I is* 1*-generic, and it is said to be* **totally** 1**-degenerate** *if $\sum_\alpha (p^\alpha)^2 \equiv 0$ on I. A* 1*-degenerate point is called a* **vertex** *of f.*

It shall be noted that, when $n \geq 3$, 1-generic curves constitute an open, dense subset (with respect to the C^∞ topology) of the space of smooth curves, either closed or not. This is a consequence of strong transversality, see [2]. On the contrary, global topological obstructions appear when the ambient space has dimension 2. For instance, the well-known four-vertex theorem ensures that every smooth, simple closed curve in $\mathbb{Q}_2$ has at least four vertexes. For an account on this result in its various forms one can consult [8], [21], [23].

Given a 1-generic curve f and a point p, by (14) we can always choose a Darboux frame with the property that $p^2 = 1$, $p^3 = \ldots = p^n = 0$, that is,

$$\phi_2^0 = \phi_0^1 = \mathrm{d}s, \qquad \phi_3^0 = \phi_4^0 = \ldots = \phi_n^0 = 0. \tag{18}$$

The isotropy subgroup for such frames is clearly

$$\text{Möb}(n)_3 = \left\{ \begin{pmatrix} 1 & x & 0 & 0 & \frac{1}{2}x^2 \\ 0 & 1 & 0 & 0 & x \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & C & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \middle| \begin{array}{c} x \in \mathbb{R} \\ C \in SO(n-2) \end{array} \right\}. \tag{19}$$

Note that, if $n = 2$, the rows and columns containing C do not appear at all and, if $n = 3$, C reduces to the real number 1.
Since this subgroup does not depend on p, we can define a **third order frame** along a 1-generic f as a second order frame satisfying (18). Now, proceeding with the reduction, we set

$$\phi_0^0 = q^2 \phi_0^1, \qquad \phi_2^b = q^b \phi_0^1 \quad \text{for } \; b \in \{3, \ldots, n\}, \; n \geq 3, \tag{20}$$

for some locally defined smooth functions q^α. Under a change of third order frame we have

$$\widetilde{q}^2 = q^2 - x, \qquad \widetilde{q}^b = C_b^d q^d, \tag{21}$$

therefore at every point $p \in I$ we can choose a third order frame such that $q^2 = 0$, hence $\phi_0^0 = 0$. The isotropy subgroup preserving such frames is

$$\text{Möb}(n)_{3\text{spec}} = \left\{ \begin{pmatrix} I_3 & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid C \in SO(n-2) \right\}, \tag{22}$$

so the reduction can be performed smoothly around every 1-generic point. We define a **special third order frame** along a 1-generic curve f as a third order frame such that

$$\phi_0^0 = 0.$$

Now the form ϕ_1^0 is independent of the chosen special third order frame so, writing

$$\phi_1^0 = \mu_1 \phi_0^1 = \mu_1 \mathrm{d}s, \tag{23}$$

defines a conformal invariant $\mu_1 \in C^\infty(I)$, which may change sign on I. We point out that the sign of μ_1 cannot be reversed by changing the (oriented)

special third order frame.
If $n = 2$ or 3, (22) reduces to the identity, and this completes the procedure. A special third order frame along a 1-generic curve f will be called a **Frenet frame**. Moreover, when $n = 3$, q^3 is a third order invariant of the generic curve f, that we shall denote by μ_2. In this case, also the sign of μ_2 may vary on I.
Assume now $n \geq 3$. The structure of the isotropy subgroup (22) (actually, even that of (19)) implies that

$$\widetilde{e}_2 = e_2, \qquad \widetilde{e}_b = C_b^c e_c \qquad b, c \in \{3, \dots, n\}, \tag{24}$$

hence N splits as the Whitney sum $< e_2 > \oplus \Theta$, where Θ is locally spanned by $\{e_b\}$ and is endowed with a natural connection, defined by setting

$$\nabla e_b = \phi_b^c \otimes e_c, \tag{25}$$

which is compatible with the Riemannian structure induced by N. We denote with $|\cdot|$ the induced norm on Θ.
By (24) and (21), the smooth section

$$X = q^b e_b \in \Gamma(\Theta) \tag{26}$$

is independent of the third order frame considered, hence globally defined. As in Definition 3.2, to be able to proceed we need a second non-degeneracy condition:

Definition 3.3. *Given an immersed curve $f : I \to \mathbb{Q}_n$, $n \geq 3$, a point $q \in I$ is called* **2-generic** *if it is 1-generic and, for a third order frame,*

$$\phi_2^c \neq 0 \quad \text{at } q \text{ for at least one } c \in \{3, \dots, n\}, \tag{27}$$

that is, $X(q) \neq 0$. The immersion f is **2-generic** *if it is 2-generic at every point, and it is* **totally 2-degenerate** *if it is 1-generic and $\phi_2^c \equiv 0$ on I for every $c \in \{3, \dots, n\}$, that is, $X \equiv 0$.*

When $n \geq 4$, for every 2-generic point $p \in I$ we can choose a special third order frame e such that X points in the direction of e_3, that is,

$$q^3 = \mu_2 > 0, \quad q^b = 0 \quad \text{for} \quad b \in \{4, \dots, n\}. \tag{28}$$

The isotropy subgroup preserving such frames has the structure

$$\text{Möb}(n)_4 = \left\{ \begin{pmatrix} I_4 & 0 & 0 \\ 0 & C_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid C_1 \in SO(n-3) \right\}. \tag{29}$$

This shows that, for 2-generic curves, we can proceed with the reduction as usual, and if $n = 4$ this is the last step. A special third order frame along a 2-generic curve f satisfying

$$\phi_2^b = 0 \quad \text{for} \quad b \in \{4, \dots, n\}.$$

will be called a **fourth order frame**. If $n = 4$ this frame will be called a **Frenet frame** since it gives the last reduction.
Summarizing what we have got so far:

- if $n = 2$, in a Frenet frame we can write the pull-back ϕ of the Maurer-Cartan form as

$$\phi = \begin{pmatrix} 0 & \mu_1 \mathrm{d}s & \mathrm{d}s & 0 \\ \mathrm{d}s & 0 & 0 & \mu_1 \mathrm{d}s \\ 0 & 0 & 0 & \mathrm{d}s \\ 0 & \mathrm{d}s & 0 & 0 \end{pmatrix}; \tag{30}$$

- if $n = 3$, in a Frenet frame we can write the pull-back ϕ of the Maurer-Cartan form as

$$\phi = \begin{pmatrix} 0 & \mu_1 \mathrm{d}s & \mathrm{d}s & 0 & 0 \\ \mathrm{d}s & 0 & 0 & 0 & \mu_1 \mathrm{d}s \\ 0 & 0 & 0 & -\mu_2 \mathrm{d}s & \mathrm{d}s \\ 0 & 0 & \mu_2 \mathrm{d}s & 0 & 0 \\ 0 & \mathrm{d}s & 0 & 0 & 0 \end{pmatrix}; \tag{31}$$

- If $n = 4$, the form ϕ^4_3 is invariant, and we can therefore write

$$\phi^4_3 = \mu_3 \phi^1_0 = \mu_3 \mathrm{d}s$$

  for some smooth function μ on I, whose sign can possibly change. The Maurer-Cartan form ϕ has the expression

$$\phi = \begin{pmatrix} 0 & \mu_1 \mathrm{d}s & \mathrm{d}s & 0 & 0 & 0 \\ \mathrm{d}s & 0 & 0 & 0 & 0 & \mu_1 \mathrm{d}s \\ 0 & 0 & 0 & -\mu_2 \mathrm{d}s & 0 & \mathrm{d}s \\ 0 & 0 & \mu_2 \mathrm{d}s & 0 & -\mu_3 \mathrm{d}s & 0 \\ 0 & 0 & 0 & \mu_3 \mathrm{d}s & 0 & 0 \\ 0 & \mathrm{d}s & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{32}$$

The previous expressions, together with the definition of ϕ, that is $\mathrm{d}e = e\phi$ give rise to the Frenet formulae

$$n = 3 \qquad\qquad\qquad\qquad n = 4$$

$$\left\{ \begin{array}{lcl} \mathrm{d}e_0 & = & \mathrm{d}s\, e_1 \\ \mathrm{d}e_1 & = & \mu_1 \mathrm{d}s\, e_0 + \mathrm{d}s\, e_4 \\ \mathrm{d}e_2 & = & \mathrm{d}s\, e_0 + \mu_2 \mathrm{d}s\, e_3 \\ \mathrm{d}e_3 & = & -\mu_2 \mathrm{d}s\, e_2 \\ \mathrm{d}e_4 & = & \mu_1 \mathrm{d}s\, e_1 + \mathrm{d}s\, e_2 \end{array} \right. \qquad \left\{ \begin{array}{lcl} \mathrm{d}e_0 & = & \mathrm{d}s\, e_1 \\ \mathrm{d}e_1 & = & \mu_1 \mathrm{d}s\, e_0 + \mathrm{d}s\, e_5 \\ \mathrm{d}e_2 & = & \mathrm{d}s\, e_0 + \mu_2 \mathrm{d}s\, e_3 \\ \mathrm{d}e_3 & = & -\mu_2 \mathrm{d}s\, e_2 + \mu_3 \mathrm{d}s\, e_4 \\ \mathrm{d}e_4 & = & -\mu_3 \mathrm{d}s\, e_3 \\ \mathrm{d}e_5 & = & \mu_1 \mathrm{d}s\, e_1 + \mathrm{d}s\, e_2 \end{array} \right. \tag{33}$$

The general reduction steps for $n \geq 5$ can now be carried on inductively. Let $4 \leq k \leq n-1$. For a $(k-2)$-generic curve, writing $\phi^c_{k-1} = q^c_{k-1}\phi^1_0$, $c \geq k$, and keeping in mind the isotropy subgroup (29), the vector field

$$X_{(k-1)} = q^c_{k-1} e_c, \qquad k \leq c \leq n$$

is globally defined and independent of the chosen k-th order frame (in this notation, it is convenient for the reader to rename X in (26) as $X_{(2)}$). Moreover, by construction $X_{(k-1)}$ is orthogonal to the span of $e_3, \dots, e_{k-1}$.

Definition 3.4. *Given an immersed curve $f : I \to \mathbb{Q}_n$ and an integer $4 \leq k < n$, a point $q \in I$ is called $(k-1)$-**generic** if it is $(k-2)$-generic and, for k-th order frames*

$$\phi^c_{k-1} \neq 0 \quad \text{at } q \text{ for at least one } c \in \{k, \dots, n\}, \tag{34}$$

*or, equivalently, $X_{(k-1)}(q) \neq 0$. The immersion f is $(k-1)$-generic if it is $(k-1)$-generic at every point, and is **totally** $(k-1)$-**degenerate** if it is $(k-2)$-generic and $X_{(k-1)} \equiv 0$ on I.*

For $(k-1)$-generic curves, we can choose a k-th order frame such that $X_{(k-1)}$ points in the direction of e_k at $p \in I$, that is,

$$\phi^k_{k-1} = \mu_{k-1}\phi^1_0, \quad \phi^b_{k-1} = 0 \quad \text{for } \; b \in \{k+1, \dots, n\} \tag{35}$$

and $\mu_{k-1} = \left|X_{(k-1)}\right| > 0$. The isotropy subgroup of the reduction has the form

$$\text{Möb}(n)_{k+1} = \left\{ \begin{pmatrix} I_{k+1} & 0 & 0 \\ 0 & C_{k-2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid C_{k-2} \in SO(n-k) \right\}, \tag{36}$$

and since it does not depend on the point $p \in I$, we can smoothly define a $(k+1)$-th order frame along a $(k-1)$-generic curve f as a k-th order frame along f such that

$$\phi^b_{k-1} = 0 \;\text{ for }\; b \in \{k+1, \dots, n\}.$$

Finally, for $k = n-1$, we have constructed, on an $(n-2)$-generic curve, an n-th order frame and, by (36), the reduction is complete.
The n-th order frame so constructed is called a **Frenet frame**.
The only form left untreated is the now globally defined ϕ^n_{n-1}. We set

$$\phi^n_{n-1} = \mu_{n-1}\phi^1_0, \qquad X_{(n-1)} = \mu_{n-1} e_n$$

for some $\mu_{n-1} \in C^\infty(I)$, not necessarily positive. According to the above definitions, we say that f is $(n-1)$-**generic** if $\mu_{n-1} \neq 0$ at every point of I, and **totally** $(n-1)$-**degenerate** if $\mu_{n-1} \equiv 0$.

For those who prefer working with Koszul's formalism, the above reduction procedure on the bundle Θ can be rephrased and summarized as follows: at each step k, we identify a vector $X_{(k-1)} \in \Gamma(\Theta)$ independent of the chosen frame; then, if $\left|X_{(k-1)}\right| > 0$, locally we set $e_k = X_{(k-1)}/\left|X_{(k-1)}\right|$ and we covariantly differentiate e_k along f with respect to the connection ∇ on Θ. $X_{(k)}$ is defined as the component of $\nabla e_k/\mathrm{d}s$ orthogonal to the span of $\{e_3, \ldots, e_k\}$. The non-degeneracy condition is equivalent to the non vanishing of this component. With all the genericity assumptions, we can proceed until we provide an orthonormal basis of Θ.

The Frenet-Serret equations $\mathrm{d}e = e\phi$ for an $(n-2)$-generic f read

$$\begin{cases} \mathrm{d}e_0 &= \mathrm{d}s\, e_1 \\ \mathrm{d}e_1 &= \mu_1 \mathrm{d}s\, e_0 + \mathrm{d}s\, e_{n+1} \\ \mathrm{d}e_2 &= \mathrm{d}s\, e_0 + \mu_2 \mathrm{d}s\, e_3 \\ \mathrm{d}e_k &= -\mu_{k-1}\mathrm{d}s\, e_{k-1} + \mu_k \mathrm{d}s\, e_{k+1} \qquad k \in \{3, \ldots, n-1\} \\ \mathrm{d}e_n &= -\mu_{n-1}\mathrm{d}s\, e_{n-1} \\ \mathrm{d}e_{n+1} &= \mu_1 \mathrm{d}s\, e_1 + \mathrm{d}s\, e_2. \end{cases} \tag{37}$$

The following characterization of degeneracy can be proved. Although this result already appears in [29], we present here a slightly different proof of group-theoretical nature.

Proposition 3.1. *Let $I \subset \mathbb{R}$ be an interval and $f : I \to \mathbb{Q}_n$, $n \geq 3$, be an immersion. Then f is totally 1-degenerate if and only if there exists a conformal circle $\mathbb{Q}_1 \subset \mathbb{Q}_n$ such that $f(I) \subset Q_1$.*
Moreover, for every $k \in \{2, \ldots, n-1\}$ if f is a $(k-1)$-generic curve, then f is totally k-degenerate if and only if there exists a conformal k-sphere $\mathbb{Q}_k \subset \mathbb{Q}_n$ such that $f(I) \subset \mathbb{Q}_k$.

Proof. We assume $k \geq 2$, the other case being analogous. Fix the index convention

$$\eta, \nu \in \{2, \ldots, k\}, \qquad b, c \in \{k+1, \ldots, n\}.$$

Consider on $\mathrm{Möb}(n)$ the ideal $\mathcal{I}$ generated by the forms $\Phi^b_0, \Phi^b_1, \Phi^b_\eta, \Phi^0_b$. Using the structure equations (4) and the symmetries of $\mathfrak{möb}(n)$, $\mathcal{I}$ can be proved to be a differential ideal, that is, $\mathrm{d}\mathcal{I} \subset \mathcal{I}$. The distribution Δ defined by $\mathcal{I}$ is therefore integrable. Moreover, at the identity, Δ is given by

$$\Delta_I = \left\{ \begin{pmatrix} a & {}^t x & 0 & 0 \\ y & D & 0 & x \\ 0 & 0 & E & 0 \\ 0 & {}^t y & 0 & -a \end{pmatrix} \middle| \begin{array}{c} D \in \mathfrak{o}(k) \\ E \in \mathfrak{o}(n-k) \\ x, y \in \mathbb{R}^k \\ a \in \mathbb{R} \end{array} \right\} \tag{38}$$

and it is obtained at any other point by left translation because of the left invariance of Φ. In particular, its maximal integral submanifold passing through the identity is the following subgroup of Möb(n):

$$T = \left\{ \begin{pmatrix} a & {}^t z & 0 & b \\ x & A & 0 & y \\ 0 & 0 & B & 0 \\ c & {}^t w & 0 & d \end{pmatrix} \middle| \begin{pmatrix} a & {}^t z & b \\ x & A & y \\ c & {}^t w & d \end{pmatrix} \in \text{Möb}(k),\ B \in SO(n-k) \right\} \simeq$$
$$\simeq \text{Möb}(k) \times SO(n-k).$$

We denote by τ, ξ the projections

$$\tau : T \to \text{Möb}(k), \qquad \xi : T \to SO(n-k).$$

For every other leaf Σ of the distribution, there exists a constant element $G \in \text{Möb}(n)$ such that $L_G(\Sigma) = T$. Moreover, the intersection $T \cap \text{Möb}(n)_0$ is isomorphic to $\text{Möb}(k)_0 \times SO(n-k)$. This implies that the following diagram commutes, where π_n, π_k are the projections that define $\mathbb{Q}_n$ and $\mathbb{Q}_k$ and $[\tau \times \xi]$ is the naturally defined quotient map:

$$\begin{tikzcd} \Sigma \arrow[r, "L_G", "\sim"'] \arrow[d, "\pi_n"] & T \arrow[r, "\tau\times\xi", "\sim"'] \arrow[d, "\pi_n"] & \text{Möb}(k) \times SO(n-k) \arrow[d, "\pi_k \times 0"] \\ \pi_n(\Sigma) \arrow[r, "G\cdot", "\sim"'] & \pi_n(T) \arrow[r, "{[\tau\times\xi]}", "\sim"'] & \mathbb{Q}_k \end{tikzcd} \tag{39}$$

From the diagram we deduce that each leaf $\pi_n(\Sigma)$ of the distribution $\pi_{n*}\Delta$ on $\mathbb{Q}_n$ is conformally equivalent to a conformal k-sphere. Let now f be totally k-degenerate. Then, in a frame e of suitable order,

$$0 = \phi_0^b = \phi_1^b = \phi_\eta^b = \phi_b^0, \tag{40}$$

hence $e_* TI$ is a subset of the distribution Δ. It follows that $e(I) \subset \Sigma$ for some leaf Σ, hence $f(I) = (\pi_n \circ e)(I) \subset \pi_n(\Sigma)$.
Since $\tau \times \xi$ is a Lie group isomorphism, it is easy to check that the conformal invariants (as well as every other conformal property) of f seen as a curve in $\mathbb{Q}_k$ are the same as those of f seen as a totally k-degenerate curve in $\mathbb{Q}_n$. Up to a conformal motion G, we can thus assume $e(I) \subset T$. Since T leaves the Lorentzian $(k+2)$-subspace $< \eta_0, \eta_1, \ldots, \eta_k, \eta_{n+1} >$ invariant and is invertible, it follows that $< e_0, e_1, \ldots, e_k, e_{n+1} > \equiv < \eta_0, \eta_1, \ldots, \eta_k, \eta_{n+1} >$ and projects to $\mathbb{Q}_k$.
Conversely, if $f(I)$ is a subset of some conformal k-sphere (that is, the projectivization of some coset of T), up to a conformal motion we can assume that $f(I) \subset \pi(T)$. Let $p \in I$. If $\sigma : \mathbb{Q}_k \to T$ is a local section around $f(p)$, then $e = \sigma \circ f$ is a local frame around p, whose form $\phi = e^{-1}de$ trivially satisfies (40). This shows that f is totally k-degenerate. $\square$

As Proposition 3.1 suggests, when f is $(k-1)$-generic we might expect that there is a unique k-sphere whose contact with the curve is at least of order k; moreover, the above proof hints that, for k-th order frames, this k-sphere be the one that comes from the linear subspace of $\mathbb{R}^{n+2}$ spanned by$\{e_0, e_1, \ldots, e_k, e_{n+1}\}$. However, classically the osculating sphere of order k is defined as the projectivization of the following subspace:

$$V_k(t) =< e_0(t), \dot{e}_0(t), \ddot{e}_0(t), \ldots, e_0^{(k+1)}(t) > \quad \forall\ 1 \leq k \leq n,$$

where the dot stands for usual derivation with respect to the parameter t. From this definition, it is not even clear whether this space has the right dimension or not. However, from $\mathrm{d}e = e\phi$ for a k^{th} order frame, it is easy to prove that

$$V_k(t) \equiv< e_0(t), e_1(t), \ldots, e_k(t), e_{n+1}(t) > . \tag{41}$$

The projectivization of the intersection of this Minkowski subspace with the light cone will be called the **osculating sphere of order** k at $f(t)$.
We recall here the Cartan-Darboux existence and uniqueness theorem for curves, stating that an immersed curve is completely characterized, up to a conformal diffeomorphism, by its conformal invariants $\{\mathrm{d}s, \mu_1, \ldots, \mu_{n-1}\}$.

Theorem 3.2 ([31] Th.3.2, 4.2, 4.3; [28], p.119)**.** *Let $I \subset \mathbb{R}$ be a compact interval, $s : I \to \mathbb{R}$ be a strictly increasing smooth function and $\mu_1, \ldots, \mu_{n-1}$, $n \geq 2$, be smooth functions on I such that*

$$\mu_k > 0 \quad \text{at every point of } I, \qquad k \in \{2, \ldots, n-2\}. \tag{42}$$

Then, up to a conformal motion of $\mathbb{Q}_n$, there exists a unique $(n-2)$-generic immersed curve $f : I \to \mathbb{Q}_n$ having s (up to an additive constant) as the conformal arclength, and $\mu_1, \ldots, \mu_{n-1}$ as given conformal invariants.

4 Euclidean and conformal invariants of curves

Let $F : I \subset \mathbb{R}^n$ be a smooth curve. In this section, we describe the links between Euclidean and conformal properties of F. Through the Dirac-Weyl chart, we can view $\mathbb{R}^n$ as an open set of the conformal sphere $\mathbb{Q}_n$. If we represent the group of rigid motions $\mathbb{E}(n)$ as a subgroup of $\text{Möb}(n)$ in the following way:

$$J \ : \ (x, A) \in \mathbb{E}(n) \longmapsto \begin{pmatrix} 1 & 0 & 0 \\ x & A & 0 \\ \frac{1}{2}|x|^2 & {}^t x A & 1 \end{pmatrix} \in \text{Möb}(n),$$

$x \in \mathbb{R}^n$, $A \in SO(n)$, then the following diagram commutes:

$$\begin{tikzcd} & \mathbb{E}(n) \arrow[r, "J"] \arrow[d, "\pi_{\mathbb{E}}"] & \text{Möb}(n) \arrow[d, "\pi"] \\ I \subset \mathbb{R} \arrow[r, "F"] & \mathbb{R}^n \arrow[r, "\chi"] & \mathbb{Q}_n \end{tikzcd} .$$

where $\pi_{\mathbb{E}}$ is the projection onto the first column. We denote $f = \chi \circ F$. To every frame $E : I \to \mathbb{E}(n)$ along F, $E = (F, E_1, \ldots, E_n)$, we can associate $e = J \circ E : I \to \text{Möb}(n)$ along f. Moreover, up to the inclusion of Lie algebras $J_{*,I_n} : \mathfrak{e}(n) \to \mathfrak{möb}(n)$, the Maurer-Cartan form Ψ of $\mathbb{E}(n)$ is the pull-back of Φ through J, so $\phi = E^*\Psi \equiv e^*\Phi$, which makes the construction consistent. From now on, we identify $\mathbb{E}(n)$ with its image through J, the homogeneous space $\pi_{\mathbb{E}} : \mathbb{E}(n) \to \mathbb{R}^n$ with $\pi : J(\mathbb{E}(n)) \to \chi(\mathbb{R}^n)$, F with $f = \chi \circ F$ and E with $e = J \circ E$. We define $\mathbb{E}(n)_0$ to be the isotropy subgroup of $[\eta_0] = \chi(0)$ in $\mathbb{E}(n)$. In this framework, a frame reduction can be carried on and gives the (Euclidean) Frenet frame along $f : I \to \mathbb{R}^n$. We very briefly sketch the procedure, which is analogous to that performed in the previous section. The first reduction gives a first order frame, characterized by the condition $\phi_0^\alpha = 0$, and the subgroup of $\mathbb{E}(n)_0$ preserving first order frames is

$$\mathbb{E}(n)_1 = \left\{ \begin{pmatrix} I_2 & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & 1 \end{pmatrix} |B \in SO(n-1). \right\} \tag{43}$$

First order frames are often called **(Euclidean) Darboux frames**, and identify a well defined normal bundle $N_{\mathbb{E}}$, $N_{\mathbb{E}} =< \{e_\beta\} >$, endowed with an inner product $(\ ,\ )$ (and induced norm $\|\cdot\|$) by requiring $\{e_\beta\}$ to be orthonormal, together with a compatible connection ∇^e given by $\nabla^e e_\beta = \phi_\beta^\alpha \otimes e_\alpha$. When $n = 2$, $\mathbb{E}(n)_1$ reduces to the identity matrix, and $N_{\mathbb{E}} =< e_2 >$.

With respect to first order frames, ϕ_0^1 is globally defined and never vanishing. This gives rise (up to an additive constant) to the **Euclidean arclength** $s_e : I \to \mathbb{R}$ such that $\phi_0^1 = \mathrm{d}s_e$. Writing $\phi_1^\alpha = h^\alpha \mathrm{d}s_e$, the quantity $k = \sum_\alpha (h^\alpha)^2$ is independent of the chosen Darboux frame, and is called the **curvature** of f. It is easy to prove that $k \equiv 0$ if and only if the curve f is a segment in $\mathbb{R}^n$. Hereafter we make the non-degeneracy assumption $k \neq 0$. Then, a further smooth reduction can be made to have $h^2 = k$ and, when $n \geq 3$, $h^b = 0$ for $b \geq 3$. The isotropy subgroup preserving such frames (**Euclidean second order frames**) is

$$\mathbb{E}(n)_2 = \left\{ \begin{pmatrix} I_3 & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & 1 \end{pmatrix} |B \in SO(n-2). \right\} \tag{44}$$

Under a change $\widetilde{e} = eK$ of second order frames, e_2 is invariant and defines the principal normal vector. Moreover, when $n \geq 3$, setting

$$v^b = \phi_2^b \left(\frac{\mathrm{d}}{\mathrm{d}s_e} \right), \qquad Y = v^b e_b \quad \text{for } 3 \leq b \leq n$$

it is immediate to show that Y is globally defined and independent of the chosen second order frame. Indeed, $Y = \nabla^e e_2/\mathrm{d}s_e$. If $n = 2$ we define $Y = 0$ to avoid separating cases in the next formulas. Define also

$$Z = k' e_2 + kY = \frac{\nabla^e(ke_2)}{\mathrm{d}s_e}, \qquad \text{where } k' = \frac{\mathrm{d}k}{\mathrm{d}s_e}. \tag{45}$$

The subsequent steps can be carried on inductively. Observe that, if $n = 3$, $Y = \tau_2 e_3$ for some smooth, globally defined function $\tau_2 = \phi_2^3(\mathrm{d}/\mathrm{d}s_e)$, possibly changing sign; τ_2 (or just τ, in this case) is called the **torsion** of the curve. If $n \geq 4$, provided the 2-genericity condition $Y \neq 0$ is satisfied, a smooth reduction can be made to have

$$\phi_2^3 = \tau_2 \phi_0^1, \qquad \phi_2^b = 0 \quad \text{for } 4 \leq b \leq n,$$

namely, $Y = \tau_2 e_3$ with $\tau_2 > 0$; the isotropy subgroup preserving such frames is analogous to $\mathbb{E}(n)_2$, with $C \in SO(n-3)$. Then we focus on the set of forms ϕ_3^b, $b \geq 4$, and so on. At every step j, the non-degeneracy condition reads $\phi_{j-1}^b \neq 0$ for at least one index $b \geq j$, and it can also be formulated as

$$Y_{(j-1)} = \phi_{j-1}^b \left(\frac{\mathrm{d}}{\mathrm{d}s_e} \right) e_b \neq 0, \qquad j \leq b \leq n.$$

The Euclidean invariants $\tau_3, \ldots, \tau_{n-1}$ are defined by $Y_{(j)} = \tau_j e_{j+1}$, $j \in \{3, \ldots, n-1\}$. In analogy with Proposition 3.1, f is totally j-degenerate ($Y_{(j)} \equiv 0$) if and only if $f(I)$ is a subset of some affine j-subspace. Up to the identification through the map J, the Euclidean Frenet-Serret equations $\mathrm{d}E = E\phi$ for an n-generic curve F, $n \geq 3$, read

$$\begin{cases} \mathrm{d}F &= \mathrm{d}s_e\, E_1 \\ \mathrm{d}E_1 &= k\mathrm{d}s_e\, E_2 \\ \mathrm{d}E_2 &= -k\mathrm{d}s_e\, E_1 + \tau_2 \mathrm{d}s_e\, E_3 \\ \mathrm{d}E_j &= -\tau_{j-1}\mathrm{d}s_e\, E_{j-1} + \tau_j \mathrm{d}s_e\, E_{j+1} \qquad 3 \leq j \leq n-1 \\ \mathrm{d}E_n &= -\tau_{n-1}\mathrm{d}s_e\, E_{n-1} \end{cases} \tag{46}$$

In order to compare the Euclidean and conformal structures, we observe that a Euclidean Darboux frame e along a curve f with $k \neq 0$ is only a first order frame in the conformal sense. To obtain a conformal second order frame ($\phi_1^\alpha = 0$ for every α) we set $\widetilde{e} = eK$, where

$$K = \begin{pmatrix} 1 & 0 & {}^t y & \frac{1}{2}|y|^2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & I_{n-1} & y \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \begin{array}{l} y = \begin{pmatrix} k \\ 0 \end{pmatrix} \in \mathbb{R}^{n-1}, \quad n \geq 2, \\ y = k \quad \text{if } n = 2. \end{array}$$

The forms change as follows:

$$\begin{aligned} &\widetilde{\phi}_0^1 = \phi_0^1 = \mathrm{d}s_e, \qquad \widetilde{\phi}_\beta^\alpha = \phi_\beta^\alpha, \qquad \widetilde{\phi}_1^0 = -\frac{k^2}{2}\phi_0^1, \\ &\widetilde{\phi}_2^0 = \mathrm{d}k = k'\phi_0^1, \qquad \widetilde{\phi}_b^0 = k\phi_2^b = kv^b\phi_0^1 \quad \text{for} \quad b \geq 3. \end{aligned} \tag{47}$$

Then, from $\widetilde{\phi}_\alpha^0 = \widetilde{p}^\alpha \widetilde{\phi}_0^1$ we deduce $\widetilde{p}^2 = k'$, $\widetilde{p}^b = kv^b$. By (45), the curve is therefore 1-generic in the conformal sense if and only if

$$\|Z\|^2 = (k')^2 + k^2\|Y\|^2 \neq 0 \qquad \text{on } I. \tag{48}$$

Let $\widehat{e} = \widetilde{e}K$, where

$$K = \begin{pmatrix} r^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & r \end{pmatrix} \in \text{Möb}(n)_2,$$

for some $B \in SO(n-1)$, $r > 0$. In order to obtain a third order frame we must have $\widehat{p}^2 = 1$, $\widehat{p}^b = 0$. From (14), $1 = \sum_\alpha (\widehat{p}^\alpha)^2 = r^4 \sum_\alpha (\widetilde{p}^\alpha)^2 = r^4\|Z\|^2$, hence

$$r = \frac{1}{\sqrt{\|Z\|}} = \left[(k')^2 + k^2\|Y\|^2\right]^{-1/4}. \tag{49}$$

Moreover, denoting by B_α the columns of B, (14) implies

$$B_2 = \begin{pmatrix} r^2 k' \\ r^2 k v^b \end{pmatrix}; \tag{50}$$

roughly speaking, B_2 has the components of $Z/\|Z\|$. Using also (47), the change of gauge gives:

$$\begin{aligned} &\widehat{\phi}_0^0 = -\mathrm{d}\log r, \qquad \widehat{\phi}_1^0 = r\widetilde{\phi}_1^0 = -r\tfrac{k^2}{2}\mathrm{d}s_e, \\ &\widehat{\phi}_0^1 = r^{-1}\widetilde{\phi}_0^1 = r^{-1}\mathrm{d}s_e, \qquad (\widehat{\phi}_\beta^\alpha) = {}^tB(\widetilde{\phi}_\beta^\alpha)B + {}^tB\mathrm{d}B = {}^tB(\phi_\beta^\alpha)B + {}^tB\mathrm{d}B. \end{aligned} \tag{51}$$

Since, for third order frames, $\widehat{\phi}_0^1$ is the differential of the conformal arclength, by (49)

$$\mathrm{d}s = \sqrt{\|Z\|}\,\mathrm{d}s_e = \left[(k')^2 + k^2\|Y\|^2\right]^{1/4}\mathrm{d}s_e; \tag{52}$$

If $n = 2$, the above formula gives $\mathrm{d}s = \sqrt{|k'|}\mathrm{d}s_e$. This relation shows that the classical definition of a vertex of a closed plane curve (that is, a stationary point of the Euclidean curvature) indeed reflects a conformal property, and coincides with Definition 3.2; $\mathrm{d}s = \sqrt{|k'|}\mathrm{d}s_e$ first appeared in [19], although the name of G. Pick is also mentioned in [6] (see also [9] and [7]).
If $n = 3$, $\|Y\|^2 = \tau^2$, and (52) is again classical and well known (see [19], [29], [33], [35]). When $n > 3$ and f is 2-generic in Euclidean sense, $Y = \tau_2 e_3$

and $\mathrm{d}s = \left[(k')^2 + k^2\tau_2^2\right]^{1/4} \mathrm{d}s_e$, see [34].
By (20), we get

$$\widehat{q}^2 = -r\frac{\mathrm{d}\log r}{\mathrm{d}s_e} = -\frac{\mathrm{d}r}{\mathrm{d}s_e}. \tag{53}$$

A special third order frame ($\overline{\phi}_0^0 = 0$) is thus constructed by setting $\overline{e} = \widehat{e}K$, with

$$K = \begin{pmatrix} 1 & \widehat{q}^2 & 0 & \widehat{q}^2/2 \\ 0 & 1 & 0 & \widehat{q}^2 \\ 0 & 0 & I_{n-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{M\ddot{o}b}(n)_3.$$

This gives

$$(\overline{\phi}_\beta^\alpha) = (\widehat{\phi}_\beta^\alpha), \qquad \overline{\phi}_1^0 = \widehat{\phi}_1^0 + \widehat{q}^2\widehat{\phi}_0^0 - \frac{1}{2}(\widehat{q}^2)^2\widehat{\phi}_0^1 + \mathrm{d}(\widehat{q}^2). \tag{54}$$

For later use, if $n \geq 3$ we search for a simple expression for the set of forms (ϕ_2^b). By (47), (51) and (54), and using (50), (45) we get

$$\begin{aligned} \overline{\phi}_2^b &= \left({}^tB(\phi_\beta^\alpha)B + {}^tB\mathrm{d}B\right)_2^b \\ &= B_b^2\phi_c^2B_2^c + B_b^c\phi_2^cB_2^2 + B_b^c\phi_d^cB_2^d + B_b^2\mathrm{d}B_2^2 + B_b^c\mathrm{d}B_2^c \\ &= B_b^2\left[-r^2k\|Y\|^2 + (r^2k')'\right]\mathrm{d}s_e + \\ &\quad\ B_b^c\left[r^2k'v^c + r^2k\phi_d^c\left(\tfrac{\mathrm{d}}{\mathrm{d}s_e}\right)v^d + (r^2kv^c)'\right]\mathrm{d}s_e \\ &= (B_b^\alpha)\left(\frac{\nabla^e(Z/\|Z\|)}{\mathrm{d}s_e}\right)^\alpha \mathrm{d}s_e \end{aligned} \tag{55}$$

From now on, we assume f to be $(n-2)$-generic, so that a complete frame reduction can be provided, giving the set of invariants $\{\mu_1, \mu_2, \ldots, \mu_{n-1}\}$, possibly with $\mu_{n-1} = 0$ somewhere. It follows that f is also generic from the Euclidean point of view, whence we can also define the Euclidean invariants $\{k, \tau_2, \ldots, \tau_{n-1}\}$. We aim to relate such sets of invariants. By (23) and using (51) and (53) we obtain

$$\mu_1 = \frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}s_e}\right)^2 - \frac{r^2k^2}{2} - r\frac{\mathrm{d}^2r}{\mathrm{d}s_e^2}, \qquad r = \left[(k')^2 + k^2\tau_2^2\right]^{-1/4}. \tag{56}$$

As we shall see, the general expression for μ_j is quite complicated. In [1] the authors gave a simple expression for some conformal invariants called $\{k_j\}$, $2 \leq j \leq n-1$, in terms of the curvature radii $\{r_j\}$ of the osculating spheres. These rise from a generalization of Coxeter's inversive distance (see also [30]). However, the authors did not relate them to the set $\{\mu_j\}$. By the results of the previous section, it is easy to show that the two sets indeed coincide. Before doing this, it is convenient to recall their procedure. A pair $S, \widetilde{S}$ of j-spheres in $\mathbb{S}^n$ (or $\mathbb{R}^n$) are viewed in $\mathbb{Q}_n$ as the projectivization of suitable Lorentzian $(j+2)$-subspaces $V, \widetilde{V}$, which can be explicitly computed. The orthogonal projections $\pi : \mathbb{R}^{n+2} \to V$, $\widetilde{\pi} : \mathbb{R}^{n+2} \to \widetilde{V}$ with respect to the Lorentzian product $\langle\ ,\rangle$ are well defined, self-adjoint operators. Setting

$$I = \pi(\widetilde{V}) \subset V, \quad \widetilde{I} = \widetilde{\pi}(V) \subset \widetilde{V}, \quad J = \ker(\widetilde{\pi}_{|V}) \subset V, \quad \widetilde{J} = \ker(\pi_{|\widetilde{V}}) \subset \widetilde{V},$$

then $\pi : \widetilde{I} \to I$, $\widetilde{\pi} : I \to \widetilde{I}$ are isomorphisms, $p_1 = \pi \circ \widetilde{\pi}_{|I}$, $p_2 = \widetilde{\pi} \circ \pi_{|\widetilde{I}}$ are conjugate to each other, hence they have the same characteristic polynomial. Clearly, such polynomial is invariant under the action of the Möbius group. In particular, its trace T_j is a conformal invariant of the pair $(S, \widetilde{S})$. If $\{w_0, \dots, w_{j+1}\}$, (resp. $\{\widetilde{w}_0, \dots, \widetilde{w}_{j+1}\}$) are orthonormal bases which diagonalize the Minkowski inner product $\langle\ ,\rangle$ and $\langle w_0, w_0\rangle = -1$ (resp. $\langle \widetilde{w}_0, \widetilde{w}_0\rangle = -1$), T_j is given by

$$T_j(S, \widetilde{S}) = \langle \widetilde{w}_0, w_0\rangle^2 - \sum_{i=1}^{j+1} \langle \widetilde{w}_0, w_i\rangle^2 - \sum_{i=1}^{j+1} \langle w_0, \widetilde{w}_i\rangle^2 + \sum_{i,k=1}^{j+1} \langle w_i, \widetilde{w}_k\rangle^2$$

Now, once a generic curve $f : I \to \mathbb{Q}_n$ is given, we can associate to each $s \in I$ the Minkowski $(j+2)$-space $V_j(s)$ giving rise to the osculating j-sphere (41). Setting $T_j(s, h) = T_j(V_j(s+h), V_j(s))$ and evaluating the Taylor polynomial in h around 0, the authors in [1] define the conformal invariants k_j as

$$k_j(s) = \sqrt{-\frac{1}{2} \frac{\partial^2 T_j}{\partial h^2}\bigg|_{h=0}} \qquad 2 \le j \le n-1, \tag{57}$$

and they show ([1], p.381 Corollary 5) that

$$k_j(s) \equiv \frac{r_{j-1}(s) r_{j+1}(s) \tau_{j+1}(s)}{r_j(s)^2 \left[(k'(s))^2 + k^2(s)\tau_2(s)^2\right]^{1/4}}. \tag{58}$$

Here we prove

Proposition 4.1. *For every $2 \le j \le n-1$, the conformal invariant $\mu_j > 0$ is given by*

$$\mu_j(s) \equiv \frac{r_{j-1}(s) r_{j+1}(s) \tau_{j+1}(s)}{r_j(s)^2 \left[(k'(s))^2 + k^2(s)\tau_2(s)^2\right]^{1/4}},$$

where r_i is the radius of the osculating i-sphere.

Proof. The following set $\{w_0, w_k, w_{j+1}\} \subset \mathbb{R}^{n+2}$:

$$w_0 = \frac{e_0 + e_{n+1}}{\sqrt{2}}, \qquad w_k = e_k \quad \text{for } 1 \le k \le j, \qquad w_{j+1} = \frac{e_0 - e_{n+1}}{\sqrt{2}},$$

gives a basis for the osculating j-sphere which diagonalizes the Lorentzian inner product. Differentiating the expression for $T_j(s, h)$, using $\langle \dot{w}_a, w_b\rangle =$

$-\langle \dot{w}_b, w_a\rangle$, $\langle \ddot{w}_a, w_a\rangle = -|\dot{w}_a|^2$, $0 \le a \le n$ we get $T_j(s,0) = j+2$ and

$$\begin{aligned}
\frac{\partial T_j}{\partial h}(s,0) &= 2\langle w_0, w_0\rangle\langle \dot{w}_0, w_0\rangle - 2\textstyle\sum_{i=1}^{j+1}\langle w_0, w_i\rangle\Big(\langle \dot{w}_0, w_i\rangle \\
&\quad + \langle w_0, \dot{w}_i\rangle\Big) + 2\textstyle\sum_{i,k=1}^{j+1}\langle w_i, w_k\rangle\langle w_i, \dot{w}_k\rangle = 0 \\
\frac{\partial^2 T_j}{\partial h^2}(s,0) &= -2\langle \ddot{w}_0, w_0\rangle - 2\textstyle\sum_{i=1}^{j+1}\langle \dot{w}_0, w_i\rangle^2 + 2\textstyle\sum_{i=1}^{j+1}\langle \ddot{w}_i, w_i\rangle \\
&\quad -2\textstyle\sum_{i=1}^{j+1}\langle w_0, \dot{w}_i\rangle^2 + \textstyle\sum_{i,k=1}^{j+1} 2\langle w_i, \dot{w}_k\rangle^2 \\
&= 2|\dot{w}_0|^2 - 4\textstyle\sum_{i=1}^{j+1}\langle \dot{w}_0, w_i\rangle^2 - 2\textstyle\sum_{i=1}^{j+1}|\dot{w}_i|^2 + \textstyle\sum_{i,k} 2\langle w_i, \dot{w}_k\rangle^2 \\
&= 2\Big(|\dot{w}_0|^2 - \textstyle\sum_{i=1}^{j+1}\langle \dot{w}_0, w_i\rangle^2\Big) \\
&\quad -2\textstyle\sum_{i=1}^{j+1}\Big(|\dot{w}_i|^2 + \langle \dot{w}_i, w_0\rangle^2 - \textstyle\sum_{k=1}^{j+1}\langle \dot{w}_i, w_k\rangle^2\Big).
\end{aligned}$$

From the Frenet-Serret equations (37) we immediately deduce that, except for the term involving $\dot{w}_j$, each addendum of the two sums in the RHS is zero since $\dot{w}_0, \dot{w}_i \in < w_0, w_1, \ldots, w_{j+1} >$, $2 \le i \le j-1$; from $\dot{w}_j = -\mu_{j-1}w_{j-1} + \mu_j e_{j+1}$ we conclude

$$\frac{\partial^2 T_j}{\partial h^2}(s,0) = -2\Big(|\dot{w}_j|^2 + \langle \dot{w}_j, w_0\rangle^2 - \sum_{k=1}^{j+1}\langle \dot{w}_j, w_k\rangle^2\Big) = -2\mu_j^2$$

Hence, by (57), $\mu_j \equiv k_j$ for every $2 \le j \le n-1$; putting this together with (58), we conclude. □

With the aid of the previous constructions, we can supply a simple proof of an interesting property already shown in [8] with different methods. Let f be a closed 1-generic curve in $\mathbb{Q}_3$. Then, we define the **total twist** of f as the normalized integral

$$\mathrm{Tw}(f) = \frac{1}{2\pi}\int_I \tau\, \mathrm{d}s_e \qquad (\mathrm{mod}\ \mathbb{Z}).$$

Proposition 4.2. *Let $f : I \to \mathbb{Q}_3$ be a closed, 1-generic curve. Then,*

$$\frac{1}{2\pi}\int_I \mu_2\, \mathrm{d}s \equiv \frac{1}{2\pi}\int_I \tau\, \mathrm{d}s_e \qquad (\mathrm{mod}\ \mathbb{Z}). \tag{59}$$

Therefore, the total twist $\mathrm{Tw}(f)$ *is a conformal invariant.*

Proof. We perform most of the proof for $f : I \to \mathbb{Q}_n$, $n \ge 3$. We assume f to be generic. Then $Y = \tau_2 e_3$ with respect to the Euclidean frame e. Therefore, we can write the vector $Z/\|Z\|$ in (45) as $\cos\theta e_2 + \sin\theta e_3$ for some θ globally defined on I. Using (50), ${}^t(B_2^\alpha) = {}^t(\cos\theta, \sin\theta, 0)$, and a suitable matrix B satisfying (50) is given by

$$B = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & I_{n-3} \end{pmatrix}.$$

A computation shows that

$$\begin{aligned} \frac{\nabla^e(Z/\|Z\|)}{\mathrm{d}s_e} &= -\theta' \sin\theta e_2 + \cos\theta \frac{\nabla^e e_2}{\mathrm{d}s_e} + \theta' \cos\theta e_3 + \sin\theta \frac{\nabla^e e_3}{\mathrm{d}s_e} \\ &= -\sin\theta(\theta' + \tau_2)e_2 + \cos\theta(\theta' + \tau_2)e_3 + \tau_3 \sin\theta e_4. \end{aligned}$$

and in (55)

$$\widehat{\phi}_2^b = \left[-B_b^2 \sin\theta(\theta' + \tau_2) + B_b^3 \cos\theta(\theta' + \tau_2) + B_b^4 \tau_3 \sin\theta\right] \mathrm{d}s_e.$$

Therefore

$$\widehat{\phi}_2^3 = (\theta' + \tau_2)\mathrm{d}s_e, \qquad \widehat{\phi}_2^4 = \tau_3 \sin\theta \mathrm{d}s_e, \qquad \widehat{\phi}_2^c = 0 \quad 5 \le c \le n.$$

If $n = 3$, then $\widehat{\phi}_2^3 = \mu_2 \mathrm{d}s$ and the computation above holds with the only requirement of 1-genericity. Since f is closed, the conclusion follows by integrating $\widehat{\phi}_2^3$ over I. □

Remark 4.3. To the best of our knowledge, a first proof of the conformal invariance of the total twist appeared in [3]. A straightforward application of Proposition 4.2 is the following: if f is included in some $\mathbb{Q}_2$, then $(2\pi)^{-1} \int_I \tau \mathrm{d}s_e$ is an integer. Indeed, in this case $\mu_2 \equiv 0$. As a matter of fact, much more is true: a surface $\Sigma \subset \mathbb{R}^3$ is an Euclidean 2-sphere if and only if $\int_I \tau \mathrm{d}s_e = 0$ for every closed curve $f : I \to \Sigma$ (Scherrer theorem, [26], subsequently generalized in [25]). The "only if" part is known as Fenchel-Jacobi theorem [14], and a different proof of it appears also in ([24], Corollary 21). Fenchel-Jacobi theorem has been generalized in [37] for embedded manifolds $M^n \to \mathbb{S}^{2n} \subset \mathbb{R}^{2n+1}$ when n is odd (Theorem 10).

5 The Euler-Lagrange equations for the conformal arclength

Let $f : I \subset \mathbb{R} \to \mathbb{Q}_n$ be an immersion. In the previous section we defined a natural parameter s, called conformal arclength, for second order frames along f. If $\mathcal{D} \subset\subset I$ is a compact subinterval, the **conformal arclength functional** on $\mathcal{D}$ is the integral

$$G_{\mathcal{D}}(f) = \int_{\mathcal{D}} \mathrm{d}s = \int_{\mathcal{D}} \sqrt[4]{\sum_\alpha (p^\alpha)^2} \phi_0^1.$$

We shall study the extremal points of the conformal arclength functional, which we shall call **conformal geodesics**.
To this end we first need to introduce the concept of admissible variation, that is a smooth map $v : I \times (-\varepsilon, \varepsilon) \to \mathbb{Q}_n$ such that

- $f(\cdot, t) : I \to \mathbb{Q}_n$ is an immersion for every $t \in (-\varepsilon, \varepsilon)$;
- $v(\cdot, 0) = f$;
- $v(x, t) = f(x)$ for x outside a compact set $K \subset I$ and $t \in (-\varepsilon, \varepsilon)$.

Let i_t denote the standard inclusion $s \in I \to (s, t) \in I \times (\varepsilon, \varepsilon)$, and for simplicity write f_t instead of $v(\cdot, t)$.
Since $\mathrm{d}s$ is only $C^{0,1/2}$ around 1-degenerate points, to avoid problems we assume f to be globally 1-generic. Since $n \geq 3$, this assumption is not restrictive. Up to choosing ε sufficiently small, we can assume that every curve f_t is 1-generic.
We also need to consider frames with good properties along every immersion f_t. Therefore we define a **special third order frame along an admissible variation** v as a frame $e : U \times (-\varepsilon, \varepsilon) \to \mathbb{Q}_n$, where $U \subset I$ is an open set, such that:

- e is a frame along v, that is, $\pi \circ e = v$;
- $e(x, t) = e(x, 0)$ for every $x \in U \setminus K$, $t \in (-\varepsilon, \varepsilon)$;
- $e_t = e(\cdot, t)$ is a special third order frame along f_t.

For every $(p, t) \in U \times (-\varepsilon, \varepsilon)$, the Maurer-Cartan form $\phi = e^*\Phi$ can be decomposed as

$$\phi_{(p,t)} = \phi(t) + \Lambda(p, t)\mathrm{d}t, \tag{60}$$

where $\phi(t)$ is just $e_t^*\Phi$ and satisfies $\phi(t)\left(\frac{\partial}{\partial t}\right) = 0$, while $\Lambda(p, t)$ is a smooth $\mathfrak{m\ddot{o}b}(n)$-valued map. With respect to special third order frames along v, ϕ satisfies

$$\begin{aligned}
&\phi_0^0 = \Lambda_0^0\mathrm{d}t; \qquad \phi_0^\alpha = \Lambda_0^\alpha\mathrm{d}t;\\
&\phi_1^\alpha = \Lambda_1^\alpha\mathrm{d}t; \qquad \phi_b^0 = \Lambda_b^0\mathrm{d}t;\\
&\phi_1^0 = \mu_1\phi_0^1(t) + \Lambda_1^0\mathrm{d}t = \mu_1\phi_0^1 + (\Lambda_1^0 - \mu_1\Lambda_0^1)\mathrm{d}t;\\
&\phi_2^0 = \phi_0^1(t) + \Lambda_2^0\mathrm{d}t = \phi_0^1 + (\Lambda_2^0 - \Lambda_0^1)\mathrm{d}t;\\
&\phi_2^b = q^b\phi_0^1(t) + \Lambda_2^b\mathrm{d}t = q^b\phi_0^1 + (\Lambda_2^b - q^b\Lambda_0^1)\mathrm{d}t,
\end{aligned} \tag{61}$$

where we have fixed the index convention $a, b, c, d \in \{3, \dots, n\}$. For convenience, we set

$$\begin{aligned}
&\lambda_0^0 = \Lambda_0^0, \qquad \lambda_0^\alpha = \Lambda_0^\alpha,\\
&\lambda_1^\alpha = \Lambda_1^\alpha, \qquad \lambda_2^0 = \Lambda_2^0 - \Lambda_0^1,\\
&\lambda_b^0 = \Lambda_b^0, \qquad \lambda_2^b = \Lambda_2^b - q^b\Lambda_0^1.
\end{aligned} \tag{62}$$

Although, in literature, the possibility of constructing variations v and special type of frames e with arbitrary initial data $\{\lambda_0^\alpha(p)\}$ is widely used without proof and is a standard fact, we have found no accessible and complete proof. The question is not straightforward since, sometimes, locally defined frames are used to compute variations of functionals whose support may be, a priori, not contained in that of the frame; of course, one can take collections $\{\lambda_0^\alpha\}$ with arbitrarily small support when computing the Euler-Lagrange equations, but since the frame depends on the variation and this latter on the set $\{\lambda_0^\alpha\}$, the question whether we can assume to have the support of the variation contained in that of the frame raises a doubt. This is important when integrating, because we need to be sure that the exact terms vanish by Stokes' theorem. If the support of the variation is not contained in that of the frame, one should show that the exact terms appearing are the differential of *globally defined objects*. Checking the invariance of such expressions is often a lengthy and very complicate computation, although straightforward, so a different strategy must be used. A complete proof of the proposition below is not difficult but requires some care, and we postpone it to the Appendix.

Proposition 5.1. *For every $p \in I$ there exists an open neighbourhood U of p such that the following holds: for every collection of $(n-1)$ smooth functions $\lambda^\alpha \in C^\infty(I)$ with compact support C included in U, there exist ε sufficiently small, a variation $v : I \times (-\varepsilon, \varepsilon) \to \mathbb{Q}_n$ and a special third order frame $e : U \times (-\varepsilon, \varepsilon) \to \mathrm{Möb}(n)$ along v such that $\lambda_0^\alpha(p, 0) = \lambda^\alpha(p)$ for every $p \in U$.*

Remark 5.2. Observe that the key fact in the above proposition is that we can take the same neighbourhood U for every collection λ^α, although ε depends on the chosen collection.

Define s to be the conformal arclength parameter of the immersion $f_0 = f$, that is, $\mathrm{d}s = \phi_0^1(0)$, and set $X_t = q^b(t)e_b(t)$. Write X instead of X_0 for notational convenience. We are ready to prove the following result:

Theorem 5.3. *When $n \geq 3$, a 1-generic immersion $f : I \to \mathbb{Q}_n$ is a conformal geodesic if and only if the following Euler-Lagrange equations are satisfied:*

$$\begin{cases} \dfrac{\mathrm{d}\mu_1}{\mathrm{d}s} + \dfrac{3}{2}\dfrac{\mathrm{d}|X|^2}{\mathrm{d}s} = 0; \\[2ex] \dfrac{\nabla^2 X}{\mathrm{d}s^2} - X(|X|^2 + 2\mu_1) = 0, \end{cases} \tag{63}$$

Proof. By the fundamental theorem of the calculus of variations, it is enough to consider variations v with arbitrarily small support. Therefore, the above existence theorem applies and we can work with a global frame containing all the support of v. Moreover, since the variation is compactly supported in $\mathcal{D}$, in the decomposition

$$\phi = \phi(t) + \Lambda \mathrm{d}t,$$

the components of Λ are compactly supported in $\mathcal{D}$.
Differentiating some of the equations in (61) and using (62), the structure equations and Cartan's lemma, we get

$$\lambda^\alpha_1\phi^1_0 - \mathrm{d}\lambda^\alpha_0 - \lambda^\beta_0\phi^\alpha_\beta + \lambda^\alpha_0\phi^0_0 = f^\alpha \mathrm{d}t; \tag{64}$$

$$\lambda^0_\alpha\phi^1_0 - \mathrm{d}\lambda^\alpha_1 - \lambda^\beta_1\phi^\alpha_\beta + \lambda^\alpha_0\phi^0_1 = g^\alpha \mathrm{d}t; \tag{65}$$

$$\mathrm{d}\lambda^0_2 + \lambda^0_b\phi^2_b - \lambda^2_1\phi^0_1 + \lambda^0_2\phi^0_0 = 2\lambda^0_0\phi^1_0 + \eta^2\mathrm{d}t, \tag{66}$$

$$\mathrm{d}\lambda^0_b + \lambda^0_2\phi^b_2 + \lambda^0_c\phi^b_c - \lambda^b_1\phi^0_1 + \lambda^0_b\phi^0_0 = \lambda^b_2\phi^1_0 + \eta^b\mathrm{d}t, \tag{67}$$

for some smooth coefficients f^α, g^α, η^α, compactly supported in $\mathcal{D}$.
Since $\mathcal{D}$ is relatively compact and Ω is smooth on I, we get

$$\frac{d}{dt}\Big|_{t=0} G_{\mathcal{D}}(f_t) = \int_{\mathcal{D}} \left(\mathcal{L}_{\frac{\partial}{\partial t}}\phi^1_0(t)\right)\Big|_{t=0} = \int_{\mathcal{D}} \left(i_{\frac{\partial}{\partial t}}\mathrm{d}\phi^1_0(t) + \mathrm{d}\left(i_{\frac{\partial}{\partial t}}\phi^1_0(t)\right)\right)\Big|_{t=0} \tag{68}$$

where $|_{t=0}$ stands for i_0^*. Since $\phi^1_0(t)\left(\frac{\partial}{\partial t}\right) = 0$,

$$\frac{d}{dt}\Big|_{t=0} G_{\mathcal{D}}(f_t) = \int_{\mathcal{D}} \left(i_{\frac{\partial}{\partial t}}\mathrm{d}\phi^1_0(t)\right)\Big|_{t=0}. \tag{69}$$

Using $\phi^1_0(t) = \phi^1_0 - \Lambda^1_0\mathrm{d}t$ we compute

$$i_{\frac{\partial}{\partial t}}\left(\mathrm{d}\phi^1_0(t)\right) = i_{\frac{\partial}{\partial t}}\left(\phi^0_0\wedge\phi^1_0 - \mathrm{d}\Lambda^1_0\wedge\mathrm{d}t\right) = i_{\frac{\partial}{\partial t}}\left(\phi^0_0\wedge\phi^1_0\right) - \frac{\partial\Lambda^1_0}{\partial t}\,\mathrm{d}t + \mathrm{d}\Lambda^1_0 \tag{70}$$

and observe that the term with $\mathrm{d}t$ will vanish when restricted to $t=0$. Moreover, the differential $\mathrm{d}\Lambda^1_0\big|_{t=0}$ has compact support in $\mathcal{D}$, so its integral vanishes by Stokes' theorem. Therefore

$$\frac{d}{dt}\Big|_{t=0} G_{\mathcal{D}}(f_t) = \int_{\mathcal{D}} \left[i_{\frac{\partial}{\partial t}}\left(\phi^0_0\wedge\phi^1_0\right)\right]\Big|_{t=0} = \int_{\mathcal{D}} \left[\lambda^0_0\phi^1_0\right]_{t=0}.$$

Using (66), the integrand becomes, at $t=0$,

$$\lambda^0_0\phi^1_0 = \frac{1}{2}\left(\mathrm{d}\lambda^0_2 + \lambda^0_b\phi^2_b - \lambda^2_1\phi^0_1 + \lambda^0_2\phi^0_0\right).$$

For ease of notation, we write $\omega \equiv \eta$ to mean that the form ω differs from η by the differential of some compactly supported function, and we omit specifying the restriction to $t=0$ when it is clear from the context.

The process is now a simple integration by parts, where we get rid of the compactly supported exact forms as they appear:

$$2\lambda_0^0\phi_0^1 \equiv \lambda_b^0\phi_b^2 - \lambda_1^2\phi_1^0 = -\lambda_b^0 q^b\phi_0^1 - \lambda_1^2\phi_1^0 \qquad \text{at } t=0.$$

Substituting (65), (64) and integrating by parts, we obtain at $t=0$

$$\begin{aligned}
2\lambda_0^0\phi_0^1 \equiv &- q^b\left(\mathrm{d}\lambda_1^b + \lambda_1^2\phi_2^b + \lambda_1^c\phi_c^b - \lambda_0^b\phi_1^0\right) - \lambda_1^2\mu_1\phi_0^1\\
\equiv &\ \lambda_1^b\mathrm{d}q^b - q^bq^b\lambda_1^2\phi_0^1 - q^b\lambda_1^c\phi_c^b + q^b\lambda_0^b\mu_1\phi_0^1 - \mu_1\left(\mathrm{d}\lambda_0^2 + \lambda_0^b\phi_b^2\right)\\
\equiv &\ \lambda_1^b\mathrm{d}q^b - q^bq^b\left(\mathrm{d}\lambda_0^2 + \lambda_0^c\phi_c^2\right) - q^b\lambda_1^c\phi_c^b + q^b\lambda_0^b\mu_1\phi_0^1\\
&+ \lambda_0^2\mathrm{d}\mu_1 + \lambda_0^b\mu_1q^b\phi_0^1\\
\equiv &\ \lambda_1^b\mathrm{d}q^b + \lambda_0^2\mathrm{d}(q^bq^b) + (q^bq^b)\lambda_0^cq^c\phi_0^1 - q^c\lambda_1^b\phi_b^c\\
&+ 2q^b\lambda_0^b\mu_1\phi_0^1 + \lambda_0^2\mathrm{d}\mu_1\\
= &\ \lambda_1^b\left(\mathrm{d}q^b + q^c\phi_c^b\right) + \lambda_0^2\mathrm{d}(q^bq^b) + (q^bq^b)q^c\lambda_0^c\phi_0^1\\
&+ 2q^b\lambda_0^b\mu_1\phi_0^1 + \lambda_0^2\mathrm{d}\mu_1.
\end{aligned}$$

Since $\phi_0^1 = \mathrm{d}s$ at $t=0$, we can write $\mathrm{d}\mu_1 = \frac{\mathrm{d}\mu_1}{\mathrm{d}s}\phi_0^1$. Moreover, recalling the definition of X, we have

$$q^bq^b = |X|^2, \qquad \mathrm{d}q^b + q^c\phi_c^b = (\nabla X)^b\,\phi_0^1.$$

Thus we have

$$\begin{aligned}
2\lambda_0^0\phi_0^1 \equiv &\,(\nabla X)^b\left(\mathrm{d}\lambda_0^b + \lambda_0^2\phi_2^b + \lambda_0^c\phi_c^b\right) + \lambda_0^2\mathrm{d}|X|^2\\
&+ |X|^2q^c\lambda_0^c\phi_0^1 + 2q^b\lambda_0^b\mu_1\phi_0^1 + \lambda_0^2\frac{\mathrm{d}\mu_1}{\mathrm{d}s}\phi_0^1\\
\equiv &- \lambda_0^b\left(\mathrm{d}(\nabla X)^b + (\nabla X)^c\phi_c^b\right) + (\nabla X)^bq^b\lambda_0^2\phi_0^1 + \lambda_0^2\mathrm{d}|X|^2\\
&+ |X|^2q^c\lambda_0^c\phi_0^1 + 2q^b\lambda_0^b\mu_1\phi_0^1 + \lambda_0^2\frac{\mathrm{d}\mu_1}{\mathrm{d}s}\phi_0^1.
\end{aligned}$$

Noting that

$$\begin{aligned}
&\mathrm{d}|X|^2 = \frac{\mathrm{d}|X|^2}{\mathrm{d}s}\phi_0^1, \qquad (\nabla X)^bq^b = \langle\nabla X, X\rangle = \tfrac{1}{2}\mathrm{d}|X|^2,\\
&\mathrm{d}(\nabla X)^b + (\nabla X)^c\phi_c^b = (\nabla^2X)^b\phi_0^1,
\end{aligned}$$

the RHS becomes

$$\left[\lambda_0^2\left(\frac{3}{2}\frac{\mathrm{d}|X|^2}{\mathrm{d}s} + \frac{\mathrm{d}\mu_1}{\mathrm{d}s}\right) + \lambda_0^b\left(-(\nabla^2X)^b + |X|^2q^b + 2q^b\mu_1\right)\right]\phi_0^1.$$

By the arbitrariness of $\lambda_0^\alpha(p,0)$, and since

$$\frac{\nabla^2X}{\mathrm{d}s^2} = (\nabla^2X)^be_b,$$

the Euler-Lagrange equations of the conformal geodesics are (63), as required. □

We now go deeper in investigating the solutions of equations (63). First, we observe that $\mathrm{d}e = e\phi$ for a special third order frame applied to the vector field $\frac{\mathrm{d}}{\mathrm{d}s}$ read

$$\begin{array}{lll} \dot{e}_0 = e_1; & \dot{e}_1 = \mu_1 e_0 + e_{n+1}; & \dot{e}_{n+1} = \mu_1 e_1 + e_2; \\ \dot{e}_2 = e_0 + X; & \dot{e}_b = -q^b e_2 + \phi_b^c \left(\frac{\mathrm{d}}{\mathrm{d}s}\right) e_c, & \end{array} \tag{71}$$

where the dot denotes the derivative of the components with respect to the parameter s. In other words, we consider the vector bundle $W =< e_0, e_A, e_{n+1} >$ associate to the principal bundle $\mathrm{Möb}(n) \to \mathbb{Q}_n$, endowed with the Lorentzian metric and we see Θ as a vector subbundle of W, with the induced (Riemannian) metric and a compatible connection ∇. From the above equations we get

$$\dot{X} = \dot{q}^b e_b + q^b \dot{e}_b = \left(\dot{q}^b + q^c \phi_c^b \left(\tfrac{\mathrm{d}}{\mathrm{d}s}\right)\right) e_b - q^b q^b e_2 = \frac{\nabla X}{\mathrm{d}s} - |X|^2 e_2. \tag{72}$$

Differentiating once more, we get

$$\dot{\overbrace{\left(\frac{\nabla X}{\mathrm{d}s}\right)}} = \frac{\nabla^2 X}{\mathrm{d}s^2} - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}|X|^2 e_2. \tag{73}$$

Define

$$V(s) =< e_0(s), e_1(s), e_2(s), e_{n+1}(s), X(s), \frac{\nabla X}{\mathrm{d}s}(s) >, \tag{74}$$

and observe that $4 \leq \dim V(s) \leq 6$, and that $V(s)$ is a Lorentzian subspace. We can prove the following result:

Theorem 5.4. *Let $f : I \to \mathbb{Q}_n$ be a 1-generic conformal geodesic. Then, $V(s)$ is a time-like vector space independent of s, which we call V. Moreover, V identifies a conformal sphere of dimension $\dim V - 2$ containing the whole immersion f.*

Proof. Choose a special third order frame along f and write $E(s)$ for the $(n+2) \times 6$ matrix $\left(e_0|e_1|e_2|X|\frac{\nabla X}{\mathrm{d}s}|e_{n+1}\right)$. Note that $\mathrm{Rank}(E) \geq 4$. Since f is a conformal geodesic, from (73) we obtain

$$\dot{\overbrace{\left(\frac{\nabla X}{\mathrm{d}s}\right)}} = X(|X|^2 + 2\mu_1) - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}|X|^2 e_2. \tag{75}$$

Integrating the first Euler-Lagrange equation we get

$$|X|^2 = -\frac{2}{3}\mu_1 + C_1, \tag{76}$$

for some constant $C_1 \in \mathbb{R}$. Using (71), (72), and (76) we can see that $E(s)$ satisfies

$$\dot{E}(s) = E(s)A(s), \qquad \text{where} \tag{77}$$

$$A(s) = \begin{pmatrix} 0 & \mu_1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & \mu_1 \\ 0 & 0 & 0 & \frac{2}{3}\mu_1 - C_1 & \frac{1}{3}\frac{\mathrm{d}\mu_1}{\mathrm{d}s} & 1 \\ 0 & 0 & 1 & 0 & \frac{4}{3}\mu_1 + C_1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{78}$$

Now, (77) is a linear system with matrix A independent of E. By the existence-uniqueness theorem for linear ODEs, the linear independence of the span of the columns of E is preserved, hence the rank of E is constant along I. Moreover, equation $\dot{E}(s) = E(s)A(s)$ implies that $V(s)$ is a vector space V independent of s. It follows that the intersection of V with the positive light cone projects to a conformal sphere of dimension $2 \leq \dim V - 2 \leq 4$ containing $[e_0]$. This concludes the proof. $\square$

Theorem 5.5. *Every conformal geodesic $f : I \to \mathbb{Q}_n$ is included in some conformal 4-sphere $\mathbb{Q}_4 \subset \mathbb{Q}_n$.*

The following theorem examines each of the three possible values taken by $\dim V$.

Theorem 5.6. *Let $f : I \to \mathbb{Q}_n$ be a 1-generic conformal geodesic.*

(i) if $\dim V = 4$, then f is a totally 2-degenerate curve of constant curvature μ_1;

(ii) if $\dim V = 5$, then f is a 2-generic and totally 3-degenerate curve. Choosing a fourth order frame, the equations satisfied by the two curvatures μ_1 and μ_2 are

$$\begin{cases} \dot{\mu}_1 + 3\mu_2\dot{\mu}_2 = 0 \\ \ddot{\mu}_2 = \mu_2^3 + 2\mu_1\mu_2, \end{cases} \tag{79}$$

where the dot denotes the derivative with respect to the arclength parameter;

(iii) if $\dim V = 6$, then f is a 3-generic and totally 4-degenerate curve. For a fifth order frame, the curvatures μ_1, μ_2, μ_3 satisfy the system

$$\begin{cases} \dot{\mu}_1 + 3\mu_2\dot{\mu}_2 = 0 \\ \ddot{\mu}_2 = \mu_2^3 + 2\mu_1\mu_2 + \mu_2\mu_3^2 \\ 2\dot{\mu}_2\mu_3 + \mu_2\dot{\mu}_3 = 0. \end{cases} \tag{80}$$

Conversely, condition $\mu_1 = \text{constant}$ *characterizes totally* 2*-degenerate conformal geodesics, a solution* $\{\mu_1, \mu_2\}$ *of* (79) *with* $\mu_2 \neq 0$ *for every* $s \in I$ *characterizes* 2*-generic and totally* 3*-degenerate conformal geodesics in* $\mathbb{Q}_3$ *and a solution of* (80) *with* $\mu_2, \mu_3 \neq 0$ *for every* $s \in I$ *characterizes* 3*-generic conformal geodesics on* $\mathbb{Q}_4$.

Proof. Observe that if $X(p) \neq 0$ for some p then, by its very definition, X is linearly independent of e_0, e_1, e_2, e_{n+1}. The same holds for $\nabla X/\mathrm{d}s$. By Proposition 3.1 and Theorem 5.4, $\dim V = 4$ if and only if both X and $\nabla X/\mathrm{d}s$ vanish identically. In this case, the first Euler-Lagrange equation becomes μ_1 constant.
In case $\dim V = 5$, then the curve is 2-generic on the whole I and totally 3-degenerate, so that $X(t) \neq 0$ for every $t \in I$. Taking a fourth order frame,

$$X = \mu_2 e_3 \quad \text{with } \mu_2 > 0 \text{ on } I; \qquad \frac{\nabla e_3}{\mathrm{d}s} = \phi_3^b\Big(\frac{\mathrm{d}}{\mathrm{d}s}\Big)e_b = 0,$$

where the last equality follows from the totally 3-degeneracy ($\phi_3^b = 0$). Differentiating, we obtain $\nabla^2 X/\mathrm{d}s^2 = \ddot{\mu}_2 e_3$, and the pair of Euler-Lagrange equations (79) are readily obtained.
In case $\dim V = 6$, the curve is 3-generic but totally 4-degenerate ($\phi_4^c = 0$ for every $c \geq 5$), so that in a fifth order frame

$$\begin{aligned}
&X = \mu_2 e_3 \quad \text{with } \mu_2 > 0 \text{ on } I;\\
&\frac{\nabla e_3}{\mathrm{d}s} = \phi_3^b\Big(\frac{\mathrm{d}}{\mathrm{d}s}\Big)e_b = \mu_3 e_4 \quad \text{with } \mu_3 > 0 \text{ on } I;\\
&\frac{\nabla e_4}{\mathrm{d}s} = \phi_4^3\Big(\frac{\mathrm{d}}{\mathrm{d}s}\Big)e_3 + \phi_4^c\Big(\frac{\mathrm{d}}{\mathrm{d}s}\Big)e_c = -\mu_3 e_3.
\end{aligned}$$

Differentiating, we get

$$\frac{\nabla^2 X}{\mathrm{d}s^2} = \left[\ddot{\mu}_2 - \mu_2\mu_3^2\right] e_3 + \left[2\dot{\mu}_2\mu_3 + \mu_2\dot{\mu}_3\right] e_4,$$

from which formulas (80) follow at once. The converse is immediate and follows from the Cartan-Darboux rigidity theorem 3.2. □

6 Integration of the equations of motion

First of all, we observe that in case the curve is totally 3-degenerate, (79) coincides with the system in [20]. We therefore limit ourselves to considering the 3-generic case

$$\begin{cases} \dot{\mu}_1 + 3\mu_2\dot{\mu}_2 = 0 \\ \ddot{\mu}_2 = \mu_2^3 + 2\mu_1\mu_2 + \mu_2\mu_3^2 \\ 2\dot{\mu}_2\mu_3 + \mu_2\dot{\mu}_3 = 0 \end{cases} \tag{81}$$

on a subset $I \subset \mathbb{R}$. Integrating the first and third equation, remembering that $\mu_2, \mu_3 > 0$ and substituting into the second one we get

$$\begin{cases} \mu_1 = -\dfrac{3}{2}\mu_2^2 + C_1, & C_1 \in \mathbb{R} \\ \mu_2^2\mu_3 = C_2, & C_2 \in \mathbb{R},\ C_2 \neq 0 \\ \ddot{\mu}_2 = -2\mu_2^3 + 2C_1\mu_2 + \dfrac{C_2^2}{\mu_2^3}, & \end{cases} \tag{82}$$

therefore the constancy of any of the curvatures implies that the geodesic f has all the curvatures constant. Since every solution of (81) is real analytic, μ_2 has either isolated stationary points or it is constant. *From now on, we assume that each curvature is not constant.* In this case, multiplying the second equation by $\dot{\mu}_2$ and integrating we obtain an equivalent differential equation, expressing a conservation of energy:

$$\frac{1}{2}\dot{\mu}_2^2 + \frac{1}{2}\mu_2^4 - C_1\mu_2^2 + \frac{C_2^2}{2\mu_2^2} = \frac{C_3}{2}, \tag{83}$$

for some $C_3 \in \mathbb{R}$. We are eventually led to solve (83) for a positive function μ_2 and an admissible triplet of real constants C_1, C_2, C_3. Multiplying by μ_2^2, taking square roots and changing variables we get

$$\int_{\mu_2^2(s_0)}^{\mu_2^2(s)} \frac{\mathrm{d}t}{\sqrt{-t^3 + 2C_1t^2 + C_3t - C_2^2}} = s - s_0. \tag{84}$$

This can be solved by using elliptic functions. Denote with ξ_-, ξ_1, ξ_2 the complex roots of the polynomial

$$P(t) = -t^3 + 2C_1t^2 + C_3t - C_2^2, \tag{85}$$

and note that $\xi_-\xi_1\xi_2 = -C_2^2 < 0$, thus only one of the following cases can happen:

$$\begin{array}{ll} (i) & \xi_- \in \mathbb{R},\ \xi_- < 0,\ \xi_1 = \bar{\xi}_2 \in \mathbb{C}\backslash\mathbb{R}; \\ (ii) & \xi_-, \xi_1, \xi_2 \in \mathbb{R},\ \xi_- < \xi_1 \leq \xi_2 < 0; \\ (iii) & \xi_-, \xi_1, \xi_2 \in \mathbb{R},\ \xi_- < 0 < \xi_1 = \xi_2; \\ (iv) & \xi_-, \xi_1, \xi_2 \in \mathbb{R},\ \xi_- < 0 < \xi_1 < \xi_2. \end{array} \tag{86}$$

Since the integral is between positive extremes and μ_2 is not constant, only case (iv) is possible, therefore $\mu_2^2 \in [\xi_1, \xi_2]$ is a bounded function. Hereafter, we restrict to the triplets (C_1, C_2, C_3) such that $P(t)$ has one negative and two distinct positive solutions. Up to a translation of the arclength parameter, and since the integral is finite around ξ_1, we can assume that $\mu_2^2 = \xi_1$ when $s = 0$, so that (84) becomes

$$\int_{\xi_1}^{\mu_2^2} \frac{\mathrm{d}t}{\sqrt{-(t - \xi_-)(t - \xi_1)(t - \xi_2)}} = s. \tag{87}$$

The change of variables

$$t = \xi_1 + \theta^2(\xi_2 - \xi_1)$$

brings us to the elliptic incomplete integral of first kind

$$\frac{2}{\sqrt{\xi_1 - \xi_-}} \int_0^{\sqrt{\frac{\mu_2^2 - \xi_1}{\xi_2 - \xi_1}}} \frac{d\theta}{\sqrt{1-\theta^2}\sqrt{1 + \frac{\xi_2 - \xi_1}{\xi_1 - \xi_-}\theta^2}} = s. \tag{88}$$

By ([18] p. 51) we can apply formula

$$\int_0^x \frac{dt}{\sqrt{b^2 - t^2}\sqrt{a^2 + t^2}} = \frac{1}{\sqrt{a^2 + b^2}} \operatorname{sd}^{-1}\left[\frac{x\sqrt{a^2+b^2}}{ab}, \frac{b}{\sqrt{a^2+b^2}}\right],$$

where $0 \leq x \leq b,\ a > 0$, with the suitable choices to obtain

$$\mu_2 = \sqrt{\xi_1 + \frac{(\xi_2 - \xi_1)(\xi_1 - \xi_-)}{\xi_2 - \xi_-}\left(\operatorname{sd}\left[s\sqrt{\frac{\xi_2 - \xi_-}{\xi_1 - \xi_-}}, \sqrt{\frac{\xi_2 - \xi_1}{\xi_2 - \xi_-}}\right]\right)^2} \tag{89}$$

We summarize the result in the following

Proposition 6.1. *There exists a non-constant solution* (μ_1, μ_2, μ_3), $\mu_2 > 0$, $\mu_3 > 0$, *of the system*

$$\begin{cases} \mu_1 = -\frac{3}{2}\mu_2^2 + C_1 \\ \mu_3 = C_2/\mu_2^2, \quad C_2 \neq 0 \\ \frac{1}{2}\dot{\mu}_2^2 + \frac{1}{2}\mu_2^4 - C_1\mu_2^2 + \frac{C_2^2}{2\mu_2^2} = \frac{C_3}{2}, \end{cases} \tag{90}$$

if and only if (C_1, C_2, C_3) *is an admissible triplet, that is,*

$$P(t) = -t^3 + 2C_1t^2 + C_3t - C_2^2 \tag{91}$$

has real roots $\xi_- < 0 < \xi_1 < \xi_2$. *In such case,* μ_2 *is given by the formula*

$$\mu_2 = \sqrt{\xi_1 + \frac{(\xi_2 - \xi_1)(\xi_1 - \xi_-)}{\xi_2 - \xi_-}\left(\operatorname{sd}\left[s\sqrt{\frac{\xi_2 - \xi_-}{\xi_1 - \xi_-}}, \sqrt{\frac{\xi_2 - \xi_1}{\xi_2 - \xi_-}}\right]\right)^2}. \tag{92}$$

Once the curvatures are known, we can even provide an explicit expression for the conformal geodesics, that is, we can write down and integrate the equation of motion for f. The case $n = 2$ is trivial and $n = 3$ already appears in [20], whose method we will follow closely. Therefore, we assume $n = 4$ and f to 3-generic, so that ϕ is given by (32). The key step is to provide a matrix $\Theta \in \mathfrak{m\ddot{o}b}(4)$, depending on the conformal curvatures, such that the system (80) defining the conformal geodesics is equivalent to the differential equation

$$\dot{\Theta} = [\Theta, \phi\left(\tfrac{\mathrm{d}}{\mathrm{d}s}\right)]. \tag{93}$$

With some computation, we find that

$$\Theta = \begin{pmatrix} 0 & 1 & -\mu_1 - \mu_2^2 & \dot{\mu}_2 & \mu_2\mu_3 & 0 \\ 0 & 0 & 0 & -\mu_2 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & -\mu_1 - \mu_2^2 \\ 0 & \mu_2 & 0 & 0 & 0 & \dot{\mu}_2 \\ 0 & 0 & 0 & 0 & 0 & \mu_2\mu_3 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \tag{94}$$

satisfies (93). Therefore, if e is the Frenet frame of the conformal geodesic, $e\Theta e^{-1}$ does not depend on s and defines a fixed element $\omega \in \mathfrak{m\ddot{o}b}(4)$. Substituting (82), a straightforward calculation yields the characteristic polynomial of Θ:

$$\chi_\Theta(t) = t^6 + 2C_1t^4 - (1 + C_3)t^2 - C_2^2, \tag{95}$$

whose roots can be computed algebraically in dependence of C_1, C_2, C_3. Note that, when $\mu_3 \equiv 0$ (i.e. $C_2 \equiv 0$), this coincides with t times the characteristic polynomial in [20]. Writing $e\Theta = \omega e$ by columns, we get the system

$$\left\{\begin{array}{lllllll} \omega e_0 & = & e_2; & \quad & \omega e_5 & = & e_1 - (\mu_1 + \mu_2^2)e_2 + \dot{\mu}_2 e_3 + \mu_2\mu_3 e_4; \\ \omega e_1 & = & e_0 + \mu_2 e_3; & & \omega e_2 & = & -(\mu_1 + \mu_2^2)e_0 + e_5; \\ \omega e_4 & = & \mu_2\mu_3 e_0; & & \omega e_3 & = & \dot{\mu}_2 e_0 - \mu_2 e_1. \end{array}\right. \tag{96}$$

A repeated application of ω to the above system gives

$$\begin{aligned} e_1 \;=\;& -\frac{1}{\mu_2^2(\dot{\mu}_2^2+1)}\omega^5 e_0 - \frac{\dot{\mu}_2}{\mu_2(\dot{\mu}_2^2+1)}\omega^4 e_0 - \frac{2(\mu_1+\mu_2^2)}{\mu_2^2(\dot{\mu}_2^2+1)}\omega^3 e_0 \\ & -\frac{2\dot{\mu}_2(\mu_1+\mu_2^2)}{\mu_2(\dot{\mu}_2^2+1)}\omega^2 e_0 + \frac{1+\dot{\mu}_2^2+\mu_2^2\mu_3^2}{\mu_2^2(\dot{\mu}_2^2+1)}\omega e_0 + \frac{\dot{\mu}_2(1+\dot{\mu}_2^2+\mu_2^2\mu_3^2)}{\mu_2(\dot{\mu}_2^2+1)}e_0. \end{aligned}$$

Using (82) and recalling that $\dot{e}_0 = e_1$ we are led to the equations of motion

$$\begin{aligned} \dot{e}_0 \;=\;& -\frac{1}{\mu_2^2(\dot{\mu}_2^2+1)}\omega^5 e_0 - \frac{\dot{\mu}_2}{\mu_2(\dot{\mu}_2^2+1)}\omega^4 e_0 + \frac{\mu_2^2-2C_1}{\mu_2^2(\dot{\mu}_2^2+1)}\omega^3 e_0 \\ & +\frac{\dot{\mu}_2(\mu_2^2-2C_1)}{\mu_2(\dot{\mu}_2^2+1)}\omega^2 e_0 + \frac{\mu_2^2+\dot{\mu}_2^2\mu_2^2+C_2^2}{\mu_2^4(\dot{\mu}_2^2+1)}\omega e_0 + \frac{\dot{\mu}_2(\mu_2^2+\dot{\mu}_2^2\mu_2^2+C_2^2)}{\mu_2^3(\dot{\mu}_2^2+1)}e_0. \end{aligned} \tag{97}$$

To solve (97), we study the endomorphism M of $\mathbb{R}^6$ represented, in the basis $\{\eta_0, \eta_A, \eta_{n+1}\}$, by the matrix ω. Observe that, from (91) and (95),

$$\chi_M(t) = \chi_\Theta(t) = P(-t^2) - t^2;$$

moreover, under the assumption (86) (iv), the polynomial $P(-x)-x$ has three distinct real roots, t_+, t_1 and t_2, satisfying

$$t_2 < -\xi_2 < -\xi_1 < t_1 < 0 < -\xi_- < t_+, \tag{98}$$

thus the eigenvalues of M are

$$\lambda = \sqrt{t_+}, \quad -\lambda, \quad i\tau_1 = i\sqrt{|t_1|}, \quad -i\tau_1, \quad i\tau_2 = i\sqrt{|t_2|}, \quad -i\tau_2. \tag{99}$$

and

$$\chi_\Theta(t) = \left(t^2-\lambda^2\right)\left(t^2+\tau_1^2\right)\left(t^2+\tau_2^2\right). \tag{100}$$

The eigenvectors relative to the real eigenvalues $\pm\lambda$ can be computed via Θ and, once expressed in the moving frame $\{e_0,\dots,e_5\}$, they read

$${}^t\!\left(\lambda^2-\frac{1}{2}\mu_2^2+C_1, \frac{\pm\lambda-\mu_2\dot{\mu}_2}{\mu_2^2+\lambda^2}, \pm\lambda, \frac{\pm\lambda\dot{\mu}_2+\mu_2}{\mu_2^2+\lambda^2}, \frac{\pm C_2}{\mu_2\lambda}, 1\right).$$

One can use this explicit expression and easily check that they are light-like. We call S_1 and S_2 the 1-dimensional eigenspaces relative to λ and $-\lambda$ respectively. Then we can decompose $\mathbb{R}^6$ as $S_1\oplus S_2\oplus F$, where F is the orthogonal complement of $S_1\oplus S_2$. We observe that F is space-like and $M_{|F}$ is a skew-symmetric endomorphism of F, with eigenvalues $\pm i\tau_1$ and $\pm i\tau_2$. By standard linear algebra, $M_{|F}$ can therefore be brought to the following block-diagonal form by an orthogonal transformation:

$$\begin{pmatrix} 0 & -\tau_1 & 0 & 0 \\ \tau_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\tau_2 \\ 0 & 0 & \tau_2 & 0 \end{pmatrix}.$$

Therefore, there exists an element $A \in \text{Möb}(4)$ such that

$$A\omega A^{-1} = \begin{pmatrix} \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\tau_1 & 0 & 0 & 0 \\ 0 & \tau_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\tau_2 & 0 \\ 0 & 0 & 0 & \tau_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\lambda \end{pmatrix} \tag{101}$$

Since we are interested in solving equations (97) up to a conformal transformation of $\mathbb{Q}_4$, we can assume that ω has the form at the RHS of (101) from the start, possibly substituting e with Ae.
Now we set $h(s) = B^{-1}e_0(s)$, with

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & i & -i & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & -i & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

so that $B^{-1}\omega B$ is diagonal. Setting

$$a_0 = \frac{\dot\mu_2(\mu_2^2+\dot\mu_2^2\mu_2^2+C_2^2)}{\mu_2^3(\dot\mu_2^2+1)}, \quad a_1 = \frac{\mu_2^2+\dot\mu_2^2\mu_2^2+C_2^2}{\mu_2^4(\dot\mu_2^2+1)}, \quad a_2 = \frac{\dot\mu_2(\mu_2^2-2C_1)}{\mu_2(\dot\mu_2^2+1)}$$
$$a_3 = \frac{\mu_2^2-2C_1}{\mu_2^2(\dot\mu_2^2+1)}, \quad a_4 = -\frac{\dot\mu_2}{\mu_2(\dot\mu_2^2+1)}, \quad a_5 = -\frac{1}{\mu_2^2(\dot\mu_2^2+1)}$$

and

$$I(t) = a_4t^4 - a_2t^2 + a_0, \qquad J(t) = a_5t^5 - a_3t^3 + a_1t$$
$$\widetilde{I}(t) = a_4t^4 + a_2t^2 + a_0, \qquad \widetilde{J}(t) = a_5t^5 + a_3t^3 + a_1t,$$

the equations of motion become

$$\dot h^0 = \left[\widetilde{I}(\lambda)+\widetilde{J}(\lambda)\right]h^0; \qquad \dot h^1 = [I(\tau_1)+iJ(\tau_1)]h^1;$$
$$\dot h^2 = [I(\tau_1)-iJ(\tau_1)]h^2; \qquad \dot h^3 = [I(\tau_2)+iJ(\tau_2)]h^3;$$
$$\dot h^4 = [I(\tau_2)-iJ(\tau_2)]h^4; \qquad \dot h^5 = \left[\widetilde{I}(\lambda)-\widetilde{J}(\lambda)\right]h^5;$$

using (83), (95) and (100) we find that

$$\widetilde{I}(\lambda) = -\frac{\lambda^2\dot\mu_2}{\mu_2(\mu_2^2+\lambda^2)}+\frac{\dot\mu_2}{\mu_2}; \qquad \widetilde{J}(\lambda) = \frac{\lambda}{\mu_2^2+\lambda^2};$$
$$I(\tau_i) = -\frac{\tau_i^2\dot\mu_2}{\mu_2(\mu_2^2-\tau_i^2)}+\frac{\dot\mu_2}{\mu_2}, \qquad J(\tau_i) = \frac{\tau_i}{\mu_2^2-\tau_i^2}, \qquad i=1,2.$$

Integrating the above equalities and observing that, by (98), (99), (100) and $\mu_2^2 \in [\xi_1,\xi_2]$ we must have $\mu_2^2 \in (\tau_1^2,\tau_2^2)$, we are led to the solutions

$$e_0^0 = p_0\sqrt{\mu_2^2+\lambda^2}\exp\left(\lambda\int_{s_0}^{s}\frac{\mathrm{d}t}{\mu_2^2+\lambda^2}\right);$$
$$e_0^1 = \rho_1\sqrt{\mu_2^2-\tau_1^2}\sin\left(\tau_1\int_{s_0}^{s}\frac{\mathrm{d}t}{\tau_1^2-\mu_2^2}-\theta_1\right);$$
$$e_0^2 = \rho_1\sqrt{\mu_2^2-\tau_1^2}\cos\left(\tau_1\int_{s_0}^{s}\frac{\mathrm{d}t}{\tau_1^2-\mu_2^2}-\theta_1\right);$$
$$e_0^3 = \rho_2\sqrt{\tau_2^2-\mu_2^2}\sin\left(\tau_2\int_{s_0}^{s}\frac{\mathrm{d}t}{\tau_2^2-\mu_2^2}-\theta_2\right);$$
$$e_0^4 = \rho_2\sqrt{\tau_2^2-\mu_2^2}\cos\left(\tau_2\int_{s_0}^{s}\frac{\mathrm{d}t}{\tau_2^2-\mu_2^2}-\theta_2\right);$$
$$e_0^5 = p_5\sqrt{\mu_2^2+\lambda^2}\exp\left(-\lambda\int_{s_0}^{s}\frac{\mathrm{d}t}{\mu_2^2+\lambda^2}\right),$$

for arbitrary constants $p_0, p_5, \rho_1, \rho_2, \theta_1, \theta_2 \in \mathbb{R}$, $\rho_i \geq 0$.
The projectivization of these solutions represents a conformal geodesic if and only if e_0 is light-like, and this happens when the constants satisfy the

conditions $2p_0p_5 = \rho_1^2 - \rho_2^2$ and $2p_0p_5\lambda^2 = \rho_2^2\tau_2^2 - \rho_1^2\tau_1^2$. The most general light-like solution is therefore

$$\begin{aligned}
e_0^0 &= A\frac{\sqrt{\tau_2^2-\tau_1^2}}{\sqrt{2}}\sqrt{\mu_2^2+\lambda^2}\ \exp\left(\lambda\int_{s_0}^{s}\frac{\mathrm{d}t}{\mu_2^2+\lambda^2}\right);\\
e_0^1 &= \sqrt{\lambda^2+\tau_2^2}\sqrt{\mu_2^2-\tau_1^2}\sin\left(\tau_1\int_{s_0}^{s}\frac{\mathrm{d}t}{\tau_1^2-\mu_2^2}-\theta_1\right);\\
e_0^2 &= \sqrt{\lambda^2+\tau_2^2}\sqrt{\mu_2^2-\tau_1^2}\cos\left(\tau_1\int_{s_0}^{s}\frac{\mathrm{d}t}{\tau_1^2-\mu_2^2}-\theta_1\right);\\
e_0^3 &= \sqrt{\lambda^2+\tau_1^2}\sqrt{\tau_2^2-\mu_2^2}\sin\left(\tau_2\int_{s_0}^{s}\frac{\mathrm{d}t}{\tau_2^2-\mu_2^2}-\theta_2\right);\\
e_0^4 &= \sqrt{\lambda^2+\tau_1^2}\sqrt{\tau_2^2-\mu_2^2}\cos\left(\tau_2\int_{s_0}^{s}\frac{\mathrm{d}t}{\tau_2^2-\mu_2^2}-\theta_2\right);\\
e_0^5 &= \frac{1}{A}\frac{\sqrt{\tau_2^2-\tau_1^2}}{\sqrt{2}}\sqrt{\mu_2^2+\lambda^2}\ \exp\left(-\lambda\int_{s_0}^{s}\frac{\mathrm{d}t}{\mu_2^2+\lambda^2}\right).
\end{aligned}$$

Let us denote by $\widetilde{e}_0$ the particular solution with $A = 1$, $\theta_1 = \theta_2 = 0$. Then the general solution e_0 is obtained as $e_0 = M\widetilde{e}_0$, where

$$M = \begin{pmatrix} A & 0 & 0 & 0 & 0 & 0\\ 0 & \cos\theta_1 & -\sin\theta_1 & 0 & 0 & 0\\ 0 & \sin\theta_1 & \cos\theta_1 & 0 & 0 & 0\\ 0 & 0 & 0 & \cos\theta_2 & -\sin\theta_2 & 0\\ 0 & 0 & 0 & \sin\theta_2 & \cos\theta_2 & 0\\ 0 & 0 & 0 & 0 & 0 & A^{-1}\end{pmatrix} \in \text{Möb}(4).$$

Therefore, up to a conformal motion of $\mathbb{Q}_4$, the conformal geodesics are given by

$$\begin{aligned}
e_0^0 &= \frac{\sqrt{\tau_2^2-\tau_1^2}}{\sqrt{2}}\sqrt{\mu_2^2+\lambda^2}\ \exp\left(\lambda\int_{s_0}^{s}\frac{\mathrm{d}t}{\mu_2^2+\lambda^2}\right);\\
e_0^1 &= \sqrt{\lambda^2+\tau_2^2}\sqrt{\mu_2^2-\tau_1^2}\sin\left(\tau_1\int_{s_0}^{s}\frac{\mathrm{d}t}{\tau_1^2-\mu_2^2}\right);\\
e_0^2 &= \sqrt{\lambda^2+\tau_2^2}\sqrt{\mu_2^2-\tau_1^2}\cos\left(\tau_1\int_{s_0}^{s}\frac{\mathrm{d}t}{\tau_1^2-\mu_2^2}\right);\\
e_0^3 &= \sqrt{\lambda^2+\tau_1^2}\sqrt{\tau_2^2-\mu_2^2}\sin\left(\tau_2\int_{s_0}^{s}\frac{\mathrm{d}t}{\tau_2^2-\mu_2^2}\right);\\
e_0^4 &= \sqrt{\lambda^2+\tau_1^2}\sqrt{\tau_2^2-\mu_2^2}\cos\left(\tau_2\int_{s_0}^{s}\frac{\mathrm{d}t}{\tau_2^2-\mu_2^2}\right);\\
e_0^5 &= \frac{\sqrt{\tau_2^2-\tau_1^2}}{\sqrt{2}}\sqrt{\mu_2^2+\lambda^2}\ \exp\left(-\lambda\int_{s_0}^{s}\frac{\mathrm{d}t}{\mu_2^2+\lambda^2}\right).
\end{aligned}$$

7 Appendix

In this section we provide a somewhat detailed proof of Proposition 5.1. We refer to Section 5 for notation.

Proposition. *For every $p \in I$ there exists an open neighbourhood U of p such that the following holds: for every collection of $(n-1)$ smooth functions $\lambda^\alpha \in C^\infty(I)$ with compact support C included in U, there exist ε sufficiently small, a variation $v : I \times (-\varepsilon, \varepsilon) \to \mathbb{Q}_n$ and a special third order frame $e : U \times (-\varepsilon, \varepsilon) \to \text{Möb}(n)$ along v such that $\lambda_0^\alpha(p, 0) = \lambda^\alpha(p)$ for every $p \in U$.*

Proof. Consider the immersion $f : I \to \mathbb{Q}_n$. We fix local coordinates x on some neighbourhood U_0 of p and (w, y^α) on a neighbourhood $V_0 \supset f(U_0)$ of $f(p) \in \mathbb{Q}_n$, with the property that the local expression of f is

$$f : x \longmapsto (x, 0).$$

Any variation v can be locally expressed, at least if t is small, as

$$v(x, t) = (x, z^\alpha(x, t)), \tag{102}$$

where z^α are real-valued functions. Note that f_t is an immersion for every t. Locally around $f(p) \in \mathbb{Q}_n$, both $\{\mathrm{d}w, \mathrm{d}y^\alpha\}$ and $\left\{\psi_0^A\right\}$ (with respect to a section σ of π) are local bases of the space of 1-forms, so there exists a non singular matrix B such that

$$\psi_0^A = B_1^A \mathrm{d}w + B_\alpha^A \mathrm{d}y^\alpha.$$

Under a change of sections of $\text{Möb}(n) \to \mathbb{Q}_n$

$$\widetilde{\psi}_0^A = r^{-1} A_A^B \psi_0^B,$$

so

$$\widetilde{\psi}_0^A = \widetilde{B}_1^A \mathrm{d}w + \widetilde{B}_\alpha^A \mathrm{d}y^\alpha,$$

with

$$\widetilde{B}_C^A = r^{-1} A_A^B B_C^B.$$

Therefore we can always choose a section σ, defined on some neighbourhood $V_1 \subset V_0$ of $f(p)$, such that $B_\alpha^1 = 0$, namely the span of ψ_0^1 coincides with that of the form $\mathrm{d}w$. Since B is nonsingular, this implies that both $B_1^1 \neq 0$ and (B_β^α) is nonsingular. The subgroup of $\text{Möb}(n)$ preserving such frames is constituted by the matrices with $A_1^\alpha = 0$. Since $A \in SO(n)$, this implies $A_\alpha^1 = 0$, $A_1^1 = 1$ and the transformation laws for the matrix B are

$$\widetilde{B}_1^1 = r^{-1} B_1^1, \qquad \widetilde{B}_1^\alpha = r^{-1} A_\alpha^\beta B_1^\beta, \qquad \widetilde{B}_\beta^\alpha = r^{-1} A_\alpha^\gamma B_\beta^\gamma.$$

Now the expression of ψ_0^A is

$$\psi_0^1 = B_1^1 \mathrm{d}w, \qquad \psi_0^\alpha = B_1^\alpha \mathrm{d}w + B_\beta^\alpha \mathrm{d}y^\beta. \tag{103}$$

Define $U = U_0 \cap f^{-1}(V_0)$, and choose an arbitrary collection of smooth functions $\{\lambda^\alpha\}$ supported in some compact set $C \subset U$. Let v be a variation of the form (102) on U. Pulling back (103):

$$\phi_0^1 = \big(B_1^1 \circ v\big)\mathrm{d}x, \qquad \phi_0^\alpha = \big(B_1^\alpha \circ v\big)\mathrm{d}x + \big(B_\beta^\alpha \circ v\big)\left(\frac{\partial z^\beta}{\partial x}\mathrm{d}x + \frac{\partial z^\beta}{\partial t}\mathrm{d}t\right) \tag{104}$$

Observe that, since $B_1^1 \neq 0$ pointwise, the first equation in (104) implies that $\{\phi_0^1\}$ never vanishes. But, on the other hand, $\phi_0^A = \phi_0^A(t) + \lambda_0^A \mathrm{d}t$ so

$$\begin{cases} \lambda_0^1 = 0; \qquad \phi_0^1(t) = \big(B_1^1 \circ v\big)\mathrm{d}x \\ \big(B_1^\alpha \circ v\big)\mathrm{d}x + \big(B_\beta^\alpha \circ v\big)\dfrac{\partial z^\beta}{\partial x}\mathrm{d}x = \phi_0^\alpha(t); \\ \big(B_\beta^\alpha \circ v\big)\dfrac{\partial z^\beta}{\partial t} = \lambda_0^\alpha. \end{cases} \tag{105}$$

Restricting the third equality to $t = 0$ we get

$$\big(B_\beta^\alpha \circ f\big)\frac{\partial z^\beta}{\partial t}\Big|_{t=0} = \lambda_0^\alpha(0). \tag{106}$$

Since (B_β^α) is always nonsingular, we can define

$$z^\alpha(x,t) = t\big(B^{-1} \circ f\big)_\beta^\alpha \lambda^\beta,$$

and we observe that, by (106), $\lambda_0^\alpha(0) = \lambda^\alpha$, $z^\alpha(x,0) = 0$ and $z^\alpha(x,t) = 0$ for every $t \in (-\varepsilon,\varepsilon)$ and $x \in U \backslash C$. The second condition allows the variation v defined in (102) to be extended smoothly to the whole $I \times (-\varepsilon,\varepsilon)$ by setting $v(p,t) = f(p)$ for $p \notin U$. Moreover, e is a local zeroth order frame along v defined on the whole U and the third of (105) is satisfied with $\lambda_0^\alpha(x,t)$ defined by

$$\lambda_0^\alpha(x,t) = (B \circ v)_\beta^\alpha \big(B^{-1} \circ f\big)_\gamma^\beta \lambda^\gamma(x).$$

Now we need to perform the frame reduction for frames along variations. The procedure is almost the same as without the dependence on t, but paying attention that, step by step, the neighbourhood U on which the frame is defined be kept fixed. Under a generic change of zeroth order frames along v, ϕ_0^A change according to

$$\widetilde{\phi}_0^1 = r^{-1}\Big(R_1^1 \phi_0^1 + R_1^\beta \phi_0^\beta\Big), \qquad \widetilde{\phi}_0^\alpha = r^{-1}\Big(R_\alpha^1 \phi_0^1 + R_\alpha^\beta \phi_0^\beta\Big) \tag{107}$$

where $R \in SO(n)$. Decomposing $\phi_0^A = \phi_0^A(t) + \lambda_0^A \mathrm{d}t$ and observing that $\lambda_0^1 = 0$ by (105), we get

$$\begin{aligned}
&\widetilde{\lambda}_0^1 = r^{-1} R_1^\alpha \lambda_0^\alpha, && \widetilde{\lambda}_0^\alpha = r^{-1} R_\alpha^\beta \lambda_0^\beta. \\
&\widetilde{\phi}_0^1(t) = r^{-1}\Big(R_1^1 \phi_0^1(t) + R_1^\beta \phi_0^\beta(t)\Big), && \widetilde{\phi}_0^\alpha(t) = r^{-1}\Big(R_\alpha^1 \phi_0^1(t) + R_\alpha^\beta \phi_0^\beta(t)\Big)
\end{aligned} \tag{108}$$

Since the set $\{\phi_0^1(t), \phi_0^\beta(t)\}$ has rank 1 on U being f_t an immersion, we can choose a suitable R so that $\widetilde{\phi}_0^\alpha(t) = 0$, that is, e_t is a first order frame for every t. R is globally defined on $U \times (-\varepsilon, \varepsilon)$, as it is apparent from the linear algebra procedure involved in its construction.
Observe that the matrix (R_β^α) is nonsingular for every (x,t); indeed, let v^α be such that $R_\alpha^\beta v^\alpha = 0$, then, since $\widetilde{\phi}_0^\alpha(t) = 0$, using (107) we get

$$0 = r^{-1} v^\alpha \Big(R_\alpha^1 \phi_0^1(t) + R_\alpha^\beta \phi_0^\beta(t)\Big) = r^{-1} v^\alpha R_\alpha^1 \phi_0^1(t),$$

therefore

$$v^\alpha R_\alpha^1 \phi_0^1(t) = 0.$$

Since $\phi_0^1(t)$ does not vanish by (105), we deduce that $R_\alpha^1 v^\alpha = 0$, that is $R_\alpha^A v^\alpha = 0$, which implies $v^\alpha = 0$ by the invertibility of R. Therefore (R_β^α) is nonsingular and in particular $(\widetilde{\lambda}_0^\alpha) = 0$ if and only if $(\lambda_0^\alpha) = 0$. Since the action of the group of rotations and homotheties is transitive on $\mathbb{R}^{n-1} \backslash \{0\}$, by (108) we can perform a change of first order frames by means of a suitable globally defined matrix of the kind

$$\widetilde{R} = \begin{pmatrix} 1 & 0 \\ 0 & C \end{pmatrix}, \qquad C \in SO(n-1),$$

to smoothly restore $\widetilde{\lambda}_0^\alpha(x,0)$ to its original value $\lambda^\alpha(x)$. We now proceed with a frame reduction to the second order keeping $\lambda^\alpha(x,0)$ unaltered. Write

$$\phi^\alpha_{1\,(p,t)} = h^\alpha(p,t)\phi_0^1(t)_p + \Lambda_1^\alpha(p,t)\mathrm{d}t = h^\alpha(p,t)\phi^1_{0(p,t)} + \lambda_1^\alpha(p,t)\mathrm{d}t, \tag{109}$$

where $\lambda_1^\alpha = -h^\alpha \lambda_0^1 + \Lambda_1^\alpha$ are smooth functions satisfying $\lambda_1^\alpha(p,t) = 0$ for $p \in U \setminus C$ and $t \in (-\varepsilon, \varepsilon)$.
A change of first order frames has values in

$$\mathrm{M\ddot{o}b}(n)_1 = \left\{ \begin{pmatrix} r^{-1} & x & {}^t yB & \frac{1}{2} r\big(x^2 + |y|^2\big) \\ 0 & 1 & 0 & rx \\ 0 & 0 & B & ry \\ 0 & 0 & 0 & r \end{pmatrix} \;\middle|\; \begin{array}{l} r \in \mathbb{R}^+, \\ B \in SO(n-1), \\ x \in \mathbb{R}, y \in \mathbb{R}^{n-1} \end{array} \right\}, \tag{110}$$

Under a change of first order frames, the coefficients h^α change according to (10): $\widetilde{h}^\alpha = r B_\alpha^\beta \big(h^\beta - y^\beta\big)$. Considering the globally defined frame $e = eK$, where K has the form above with $B = I_{n-1}$ and $y^\alpha = h^\alpha$ we can pass to a second order frame on U. Moreover, by the change of λ_0^α in (108) we get $\widetilde{\lambda}_0^\alpha(x,0) = \lambda_0^\alpha(x,0) = \lambda^\alpha(x)$. The last two steps involve the coefficients p^α, defined by $\phi_\alpha^0(t) = p^\alpha \phi_0^1(t)$, and $\phi_0^0(t)$. Since the curve is 1-generic, up to choosing ε small enough we can assume that every curve of the variation is 1-generic. The procedure is then identical to the one in Section 3 and, arguing as above, it is easy to find the desired special third order frame globally defined on U. This concludes the proof. $\square$

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